HEAT EQUATION

A PROJECT REPORT

SUBMITTED BY

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UID NUMBER: 21013121003

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BANKURA UNIVERSITY

BANKURA CHRISTIAN COLLEGE

BANKURA

2023-24

HEAT EQUATION

A PROJECT REPORT

Under the supervision of Dr. Subhasis Bandyopadhyay

Submitted By

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BANKURA CHRISTIAN COLLEGE

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CERTIFICATE

This is to certify that **Trishna Nag (UID: 21013121003, Registration No.: 00330 of 2021-22)** of **Department of Mathematics, Bankura Christian College, Bankura,** has successfully carried out this project work entitled "**HEAT EQUATION**" under my supervision and guidance.

This project has been undertaken as a part of the Curriculum of **Bankura University** (Semester – VI, Paper: DSE-4 (Project Work)) (Course ID: 62117) and for the partial fulfillment for the degree of **Bachelor of Science (Honours) in Mathematics of Bankura University in 2023 – 24.**

Signature of Supervisor
Department of Mathematics
Bankura Christian College

Signature of Head of the Department
Department of Mathematics
Bankura Christian College

DECLARATION

I hereby declare that my project, titled '**Heat Equation'**, submitted by me to Bankura University for the purpose of **DSE - 4** paper, in semester VI (A.Y.: 2023 – 24) under the guidance of my teacher Dr. Subhasis Bandyopadhyay, Department of Mathematics, Bankura Christian College,

I also declare that the project has not been submitted here by any other student.

Trishna Nag

Semester: VI

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ACKNOWLEDGEMENT

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Trishna Nag

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INTRODUCTION

The heat equation is the most important example of a linear **Partial Differential Equation** of parabolic type. The theory of the heat equation was first developed by **Joseph Fourier** in **1822** for the purpose of modeling how a quantity such as heat diffuses through a given region. Fourier's work, particularly his 1822 treatise for modern mathematical physics and introduced the concept of Fourier series, which are essential in solving heat related problems. His work not only transformed the understanding of thermal phenomena but also had for reaching implications in various scientific and engineering disciplines.

"*The effects of heat are subject to constant laws which can't be discovered without the aid of mathematical analysis*," said Joseph Fourier.

The derivation of the heat equation involves applying the principle of conservation of energy to a small volume of material. Suppose we have a function $\boldsymbol{u(x, y, z, t)}$, which describes the temperature of a conducting material at a given location $\boldsymbol{(x, y, z)}$, I can use this function to determine the temperature at any position on the material at a future time. The function $\boldsymbol{u}$ changes over time as heat spreads throughout the material and the heat equation is used to determine this change in the function $\boldsymbol{u}$. The gradient of $\boldsymbol{u}$ describes which direction and at what rate is the temperature changing around a particular region of the material. Therefore, the gradient of temperature is the flow of heat through the material. This gradient will help us determine the flow of heat through various materials. This is analogous to the flow of water in a pipe.

The simplest form of the one dimensional heat equation is

$$u_t = K u_{xx}$$

where $u(x,t)$ represents the temperature distribution, t is time, K is thermal diffusivity of the material. The equation is a parabolic partial differential equation, capturing the essence of how heat diffuses through a medium. It is another classical equation of mathematical physics and it is very different from wave equation. This equation describes also a diffusion, so we sometimes will refer to it as diffusion equation.

This heat equation uses in various field such as *modeling problem of science, engineering and financial engineering, fluid mechanics, weather forecasting climate physics, geo-physics, solar physics, atmospheric scienceetc.*

IMPORTANCE

- **Fundamental Understanding:** The heat equation is a fundamental PDE that describes the distribution of heat in a given region over time. It's behavior provides various physical phenomena like heat conduction, diffusion process and thermal equilibrium.

- **Educational Importance:** Learning about heat in PDE provide us valuable mathematical and computational skills that are applicable in various scientific and engineering fields also it helps us to develop problem solving ability and gain a deeper understanding of mathematical modeling and simulation technique.

- **Medical Applications:** In medical physics and bio physics the heat equation is used to model thermal process in biological tissues, such as hyperthermia treatments for cancer therapy and thermal ablation techniques for tissue destruction.

- **Climate Science:** The heat equation is also relevant in climate science for modeling the distribution of heat in atmosphere, ocean and land surface. How heat transformed and distributed in earth system is helpful for us to study about climate change, weather patterns, environmental processes.

- **Predictive Modeling:** By solving the heat equation, engineers and scientists can predict how temperature distributions evolve over time and space in complex systems. It is essential for designing efficient and reliable thermal systems and optimizing their performance.

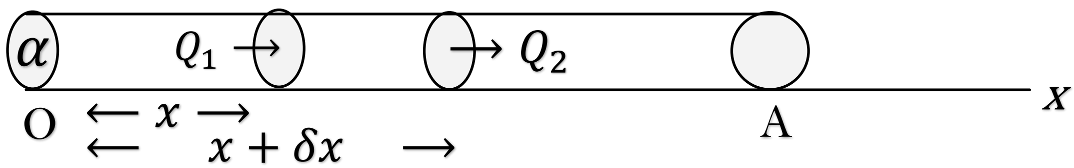
There are some more reason to study the heat equation in PDE like engineering applications designing and optimization, safety and reliability and computational modeling etc.

OBJECTIVE

My interest in this topic grow when I first read Fourier series in real analysis and wave equation in PDE. Here are some steps I shall follow to complete my project over one dimensional heat equation.

1. **Introduction to the One-Dimensional Heat Equation:** Firstly we will introduce the one-dimensional heat equation in partial differential equation that describes the distribution of heat over a one-dimensional domain.
2. **Problem Statement:** In this project, we aim to investigate the behaviour of heat conduction in a one-dimensional domain using the one-dimensional heat equation. Specifically, we have to determine the temperature distribution within a material rod subjected to certain boundary conditions and initial temperature profiles.
3. **Mathematical Formulation:** The one-dimensional heat equation is given by $u_t = c^2 u_{xx}$,where $u(x,t)$ represents the temperature distribution at position x and time t, c^2 is thermal diffusivity of the material.
4. **Analytical Methods**: To solve the one-dimensional heat equation analytically we will employ separation of variable method. In this method we will first observe and discuss the initial and boundary conditions and solve the eigenvalues and explain how the superposition principle allows for constructing the general solution by summing the individual solutions corresponding to different eigenvalues.
5. **Validation and Analysis:** We will validate the result of the analytical solutions obtained using separation of variables by comparing them with numerical simulations or experimental data if available.
6. **Conclusion**: Lastly we will summarize the key findings and insights gained from applying separation of variables to solve the one-dimensional heat equation. And discuss the limitations of the approach in modeling and solution techniques.

DERIVATION OF 1-D HEAT EQUATION



Consider a homogeneous bar of uniform cross section $\alpha\ (cm^2)$. Suppose that the size are covered with material impervious to heat. So that the stream line of heat flow are perpendicular to the area α.

Take one end of the bar as origin and the direction of flow as positive x-axis. Let ρ be the density (g/cm^3) and s be the specific heat $(cal/g \cdot deg)$ and k be the thermal conductivity $(cal/cm \cdot deg \cdot sec)$. Suppose that the bar is raised to an assigned temperature distribution at time $t = 0$and then heat is allowed to flow by conduction.

Let $u(x,t)$ be the temperature at a distance x from origin O any time t.

If δu be the temperature change in slab of thickness δx of the bar, then the quantity of heat in this slab $=\ \rho \cdot \alpha \cdot \delta x \cdot s\frac{\partial u}{\partial t}$. Where $Q_1, Q_2$are respectively the rate (cal/sec) of inflow and outflow of heat.

Also we know from Fourier law of heat conduction, if $Q(cal/sec)$ be the rate of heat flow through a slab of area $\alpha(cm^2)$ and thickness δx where the differences of temperature at the faces in δu then $Q = -P \cdot \alpha \frac{\partial u}{\partial x}$

where P is thermal conductivity.

Now,

$$Q_1 = -P\alpha \left[\frac{\partial u}{\partial x}\right]_{at\ x} \quad and \quad Q_2 = -P\alpha \left[\frac{\partial u}{\partial x}\right]_{at\ x+\delta x}$$

$$\therefore \rho \cdot \alpha \cdot \delta x \cdot s \frac{\partial u}{\partial t} = P\alpha \left[\frac{\partial u}{\partial x}\right]_{at\ x+\delta x} - P\alpha \left[\frac{\partial u}{\partial x}\right]_{at\ x}$$

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{P}{\rho s} \left\{ \frac{\left[\frac{\partial u}{\partial x}\right]_{at\ x+\delta x} - \left[\frac{\partial u}{\partial x}\right]_{at\ x}}{\delta x} \right\}$$

Writing $\frac{P}{\delta s} = K$, called the diffusivity of the substance(cm^2/sec)and taking limit as $\delta x \to 0$, we get

$$\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$$

which is the required one dimensional heat equation or one dimensional diffusion equation.

GENERAL SOLUTION OF 1-D HEAT EQUATION

❖ Using method of separation of variables

Consider $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$ ________________ (1)

Let the solution is the form $u(x,t) = X(x) \cdot T(t) \neq 0$

$$\therefore \frac{\partial u}{\partial t} = XT', \ \frac{\partial^2 u}{\partial x^2} = X''T$$

where dashes denote the derivative with respect to relevant variables.

From (1), $XT' = K \cdot X''T$

$$\Rightarrow \frac{X''}{X} = \frac{1}{K} \cdot \frac{T'}{T} \quad \text{\_\_\_\_\_\_\_\_\_\_\_ } (2)$$

As x and t are independent variables, so (2) is true if both side is equal to some constant μ, called arbitrary separation constant.

$$\therefore \frac{X''}{X} = \frac{1}{K} \cdot \frac{T'}{T} = \mu$$

$\Rightarrow X'' - \mu X = 0$ _______ (3) and $T' - \mu KT = 0$ ________ (4)

Following three cases arise :-

- **Case 1:** When $\mu = 0$

  From (3) & (4) we get,

  $X'' = 0$ and $T' = 0$

  $\Rightarrow X(x) = A_1 x + B_1$ and $T(t) = C_1$

  $\therefore u(x,t) = (A_1 x + B_1)C_1$

  $= A_2 x + A_3$

  where $A_2 = A_1 C_1$ and $A_3 = B_1 C_1$ are arbitrary constant.

- **Case 2:** When $\mu > 0$, say $\mu = \lambda^2, \ \lambda \neq 0$

  From (3) & (4) we get,

  $X'' - \lambda^2 x = 0$ and $T' - \lambda^2 KT = 0$

  $\Rightarrow X(x) = a_1 e^{\lambda x} + b_1 \mathrm{e}^{-\lambda x}$ and $T(t) = C_1 e^{\lambda^2 KT}$

$$\therefore u(x,t) = X(x)T(t)$$

$$= \left(a_1 e^{\lambda x} + b_1 \mathrm{e}^{-\lambda x}\right) C_1 e^{\lambda^2 KT}$$

$$= (A_4 e^{\lambda x} = A_5 e^{-\lambda x}) e^{\lambda^2 KT}$$

where $A_4 = a_1 C_1$ and $A_5 = b_1 C_1$ are arbitrary constant.

- **Case 3:** When $\mu < 0$, say $\mu = -\lambda^2,\ \ \lambda \neq 0$

  From (3) & (4) we get,

  $X'' + \lambda^2 x = 0$ and $T' + \lambda^2 KT = 0$

$$\Rightarrow X(x) = a_1 cos\,(\lambda x) + b_1 sin\,(\lambda x) \quad \text{and} \quad T(t) = C_2 e^{-\lambda^2 KT}$$

$$\therefore u(x,t) = \{a_2\, cos(\lambda x) + b_2 si\, n(\lambda x)\} C_2 e^{-\lambda^2 KT}$$

$$= \{A_6\, cos(\lambda x) + A_7 si\, n(\lambda x)\} e^{-\lambda^2 KT}$$

where $A_6 = a_2 C_2$ and $A_7 = b_2 C_2$ are arbitrary constant.

∴ Three possible solutions are

$$u(x,t) = A_2 x = A_3$$

$$u(x,t) = \left(A_4 e^{\lambda x} + A_5 e^{-\lambda x}\right) e^{\lambda^2 KT}$$

$$u(x,t) = \ \{A_6\, cos(\lambda x) + A_7 si\, n(\lambda x)\} e^{-\lambda^2 KT}$$

where $A_2, A_3, A_4, A_5, A_6, A_7$ are arbitrary constant.

We have to choose that solution which is consistent with the physical nature of the problem.

- **Initial and boundary condition for one dimensional heat equation:**

Consider one dimensional heat equation $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$

where $u(x,t)$ is temperature at any point x and ant time t.

The boundary conditions are,

$$\text{(i) } u(0,t) = T_1 ℃ \ \forall t$$

$$\text{(ii) } u(l,t) = T_2 ℃ \ \forall t$$

where the length of the rod or bar is l.

The initial condition is

$$u(x,0) = f(x), 0 < x < l$$

- **Steady state condition for one dimensional heat equation:**

When steady state condition arise, the temperature is independent of time

i.e. $\frac{\partial u}{\partial t} = 0$

The heat equation becomes, $\frac{\partial^2 u}{\partial x^2} = 0$ or $\frac{d^2 u}{dx^2} = 0$

- **Insulated ends:**

If an end of heat conducting bar is insulated it means no heat passes through that section.

i.e. the temperature gradient is zero at that point.

i.e. $\frac{\partial u}{\partial x} = 0$ at that point.

SOLUTION OF 1-D HEAT EQUATION

❖ For homogeneous boundary conditions

There are four types of problem for homogeneous boundary conditions we will discuss.

- Type 1: (When both the ends are at zero temperature and the initial temperature is given by f(x)).

  If both ends of a bar of length 'a' at temperature 0 and the initial temperature is prescribed by the function f(x). Then find the temperature distribution u(x,t).

Here we have to find $u(x,t)$, where $u(x,t)$ is the solution of the problem

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Subject to, boundary conditions $u(0,t) = 0 \quad \forall t \geq 0$ (2)

$u(a,t) = 0 \quad \forall t \geq 0$ (3)

initial conditions $u(x,0) = f(x) \quad 0 < x < a$ (4)

By method of separation of variables, the three possible solutions are

$$u(x,t) = A_1 x + A_2 \tag{5}$$

$$u(x,t) = \left(B_1 e^{\lambda x} + B_2 e^{-\lambda x}\right)e^{\lambda^2 Kt} \tag{6}$$

$$u(x,t) = \{\text{A}\cos(\lambda x) + B\sin(\lambda x)\}e^{-\lambda^2 Kt} \tag{7}$$

where A_1, A_2, B_1, B_2, A, B are arbitrary constants and λ is arbitrary separation constant.

Now applying conditions (2) & (3) in the solution (5) we get,

$u(0,t) = 0$ and $u(a,t) = 0$

$\Rightarrow A_2 = 0$ and $A_1 a + a_2 = 0$

Both imply, $A_1 = A_2 = 0$

From (5), $u(x,t) = 0$ which is trivial. So, we reject this solution.

Now applying conditions (2) & (3) in the solution (6) we get,

$u(0,t) = 0$

$\Rightarrow (B_1 + B_2)e^{\lambda^2 Kt} = 0$

$\therefore B_1 + B_2 = 0$

$u(a,t) = 0$

$\left(B_1 e^{\lambda a} + B_2 e^{-\lambda a}\right)e^{\lambda^2 Kt} = 0$

$B_1 e^{\lambda a} + B_2 e^{-\lambda a} = 0$ as $e^{\lambda^2 Kt} \neq 0$

Together imply, $B_1 = B_2 = 0$

From (6), $u(x,t) = 0$ which is trivial. So, we reject this solution.

Now applying (2) & (3) in the solution (7) we get,

$u(0,t) = 0$

$\Rightarrow Ae^{-\lambda^2 Kt} = 0$

$\Rightarrow A = 0$

$u(a,t) = 0$

$\Rightarrow \{Acos(\lambda a) + Bsin(\lambda a)\}e^{-\lambda^2 Kt} = 0$

$\Rightarrow Asos(\lambda a) + Bsin(\lambda a) = 0$

Together imply, $Bsin(\lambda a) = 0$

If $B = 0$ then $u(x,t) = 0$ which is trivial.

So, we take $B \neq 0$

$\therefore \sin(\lambda a) = 0$

$\Rightarrow \lambda a = n\pi, \ \ n = 1,2,3, \dots$

$\Rightarrow \lambda = \frac{n\pi}{a}, \ \ n = 1,2,3 \dots$ (8)

Hence non-zero solutions $u_n(x,t)$ are given by,

$$u_n(x,t) = A_n \left\{\cos\left(\frac{n\pi x}{a}\right) + B_n \sin\left(\frac{n\pi x}{a}\right)\right\} e^{\frac{-n^2\pi^2 Kt}{a^2}} \quad for\ n = 1,2,3, \dots$$

As equation (1) is linear and homogeneous so, by super position principal the most general solution $u(x,t)$ is given by,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi x}{a}\right) + B_n \sin\left(\frac{n\pi x}{a}\right) \right\} e^{\frac{-n^2\pi^2 K}{a^2}}$$

$$= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) e^{\frac{-n^2\pi^2 K}{a^2}t} \quad as\ \ A_n = 0\ \forall n \tag{9}$$

Now applying the initial condition $u(x, 0) = f(x)$

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right)$$

which is Fourier sine series. So, the constants are given by,

$$B_n = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi x}{a}\right) dx \tag{10}$$

Therefore, the solution of the given problem is given by (9), where B_n are given by (10).

- <u>Type 2: (Steady state condition and zero boundary condition)</u>

A rod of length 'l' has its ends A and B kept at $\mathbf{0}$°C and $\boldsymbol{T}$°C respectively until steady state condition prevail. If the temperature at B is reduced to $\mathbf{0}$°C and kept so while that of A is maintained. Find the resulting temperature distribution $\boldsymbol{u(x, t)}$ taking origin at A.

- **When steady state condition prevails:**

The heat equation becomes, $\boldsymbol{\frac{d^2u}{dx^2} = 0}$

$$\Rightarrow u = ax + b$$

where a & b are arbitrary constants.

Given that,

$u = 0$ when $x = 0$ | $u = T$ when $x = l$.

$\Rightarrow a \cdot 0 + b = 0$

$\Rightarrow b = 0$

$\Rightarrow a \cdot l + b = T$

$\Rightarrow al + b = T$

Both implying, $a = \frac{T}{l}$, $b = 0$

$\therefore \boldsymbol{u = \frac{T}{l} x}$**,** which is the solution at steady state condition.

- **After steady state condition:**

The problem is $\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}$

Subject to, $u(0, t) = u(l, t) = 0$

and $u(x, 0) = f(x) = \frac{T}{l} x, \ 0 < x < l$

$\therefore$ The solution is $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-Kn^2\pi^2 t}{l^2}}$

where $B_n = \frac{2}{l}\int_0^l \frac{T}{l} x sin\left(\frac{n\pi x}{l}\right) dx$

$$= \frac{2T}{l^2}\left\{\left[-\frac{l}{n\pi} x cos\left(\frac{n\pi x}{l}\right)\right]_0^l + \frac{l}{n\pi}\int_0^l \cos\left(\frac{n\pi x}{l}\right) dx\right\}$$

$$= \frac{2T}{l^2}\left\{\frac{l^2}{n\pi}(-1)^{n+1} + \frac{l^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{l}\right)\right]_0^l\right\}$$

$$= \frac{2T}{n\pi}(-1)^{n+1}$$

$\therefore$ Required solution is.

$$u(x,t) = \frac{2T}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-Kn^2\pi^2 t}{l^2}}$$

- <u>Type 3: (Both ends are insulated and the initial temperature is given by f(x))</u>

If both ends of a bar of length 'l' are insulated and the initial temperature f(x) is prescribed then find the temperature distribution u(x,t).

We have to solve $$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \quad (1)$$

Subject to, boundary conditions $u_x(0,t) = 0 \ \ \forall t \geq 0$ (2)

$u_x(l,t) = 0 \ \ \forall t \geq 0$ (3)

initial conditions $u(x,0) = f(x), \ 0 < x < l$ (4)

By method of separation of variables the equation (1) has three possible solutions which are ,

$$u(x,t) = A_1x + A_2 \quad (5)$$

$$u(x,t) = \left(A_3e^{\lambda x} + A_4e^{-\lambda x}\right)e^{K\lambda^2 t} \quad (6)$$

$$u(x,t) = \{A_5\cos(\lambda x) + A_6\sin(\lambda x)\}e^{-K\lambda^2 t} \quad (7)$$

where A_1, A_2, A_3, A_4, A_5, A_6 are arbitrary constants and λ is arbitrary separation constant.

Now applying conditions (2) & (3) in the solution (5) we get,

$u_x(0,t) = 0$ and $u_x(l,t) = 0$

$\Rightarrow A_1 = 0$

$\therefore u(x,t) = A_2 = \frac{1}{2}a_0$ (say) where, $a_0 = 2A_2$ is arbitrary constant.

In this case we get a solution $\boldsymbol{u_0(x,t) = \frac{1}{2}a_0}$ (8)

Now applying conditions (2) & (3) in the solution (6) we get,

$u_x(0,t) = 0$

$\Rightarrow \lambda(A_3 - A_4)e^{K\lambda^2 t} = 0$

$\Rightarrow A_3 - A_4 = 0$ as $\lambda e^{K\lambda^2 t} \neq 0$

$\Rightarrow A_3 = A_4$

$u_x(l,t) = 0$

$\Rightarrow \lambda\left(A_3e^{\lambda l} - A_4e^{-\lambda t}\right)e^{K\lambda^2 t} = 0$

$\Rightarrow A_3e^{\lambda l} - A_4e^{-\lambda t} = 0$

$\Rightarrow A_4\left(e^{\lambda l} - e^{-\lambda l}\right) = 0 \ \Rightarrow A_4 = 0$

So, in this case $u(x,t) = 0$, which is trivial. So, we reject this case.

Now applying conditions (2) & (3) in the solution (7) we get,

$u_x(0,t) = 0$

$\Rightarrow \lambda A_6 e^{-K\lambda^2 t} = 0$

$\Rightarrow A_6 = 0$ as $\lambda e^{-K\lambda^2 t} \neq 0$

$u_x(l,t) = 0$

$\Rightarrow \lambda[-A_5 \sin(\lambda l) + A_6 \cos(\lambda l)]e^{-K\lambda^2 t} = 0$

$\Rightarrow A_5 \sin(\lambda l) + A_6 \cos(\lambda l) = 0$

$\Rightarrow A_5 \sin(\lambda l) = 0$ as $A_6 = 0$

If $A_5 = 0$ then $u(x,t) = 0$ is trivial.

So, let $A_5 \neq 0$ $\quad \therefore \sin(\lambda l) = 0$

$\Rightarrow \lambda l = n\pi\,,\ n = 1,2,3,\ldots$

$\Rightarrow \lambda = \frac{n\pi}{l}\,, n = 1,2,3,\ldots$

From (7) we get the solutions are

$$u_n(x,t) = a_n \cos\left(\frac{n\pi x}{l}\right) e^{\frac{-Kn^2\pi^2 t}{l^2}}\,, n = 1,2,3,\ldots$$

As equation (1) is linear and homogeneous so by super position principal from (8) & (9) we get the most general solution is given by

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) e^{\frac{-Kn^2\pi^2 t}{l^2}} \qquad (10)$$

Now using (4) from (10) we get,

$$u(x,0) = f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \ \ in\ \ (0,l)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)dx$ (11)

and $a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx,\ n = 1,2,3,\ldots$ (12)

Therefore, the required solution of the given problem is given by (10) where a_0 is given by (11) & a_n is given by (12).

- <u>Type 4: (One end is insulated and the other end is at zero temperature and initial temperature is given by f(x))</u>

If one end of a bar of length 'l' is insulated and the other end is at temperature zero and the initial temperature f(x) is prescribed then find the temperature distribution u(x,t).

We have to solve $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$ (1)

Subject to, boundary conditions $u_x(0,t) = 0 \ \forall t \geq 0$ (2)

$u(l,t) = 0 \ \forall t \geq 0$ (3)

initial conditions $u(x,0) = f(x), \ 0 < x < l$ (4)

By method of separation of variables, the equation (1) has three possible solutions which are,

$u(x,t) = A_1 x + A_2$ (5)

$u(x,t) = \left(A_3 e^{\lambda x} + A_4 e^{-\lambda x}\right)e^{K\lambda^2 t}$ (6)

$u(x,t) = \{A_5\cos(\lambda x) + A_6\sin(\lambda x)\}e^{-K\lambda^2 t}$ (7)

where A_1, A_2, A_3, A_4, A_5, A_6 are arbitrary constants and λ is arbitrary separation constant.

Now applying conditions (2) & (3) in the solution (5) we get,

$u_x(0,t) = 0$ and $u_x(l,t) = 0$

$\Rightarrow A_1 = A_2 = 0$

$\therefore u(x,t) = 0$ which is trivial. So, we reject this solution.

Now applying conditions (2) & (3) in the solution (6) we get,

$u_x(0,t) = 0$ $\qquad$ $u(l,t) = 0$

$\Rightarrow A_3 - A_4 = 0$ $\qquad$ $\Rightarrow A_3 e^{\lambda l} + A_4 e^{-\lambda t} = 0$

Together imply, $A_3 = A_4 = 0$

In this case $u(x,t) = 0$, which is trivial. So, we reject this case.

Now applying conditions (2) & (3) in the solution (7) we get,

$u_x(0,t) = 0 \Rightarrow A_6 = 0$ and $u(l,t) = 0 \Rightarrow A_5 \cos(\lambda l) = 0$

If $A_5 = 0$ then $u(x,t) = 0$ which is trivial.

So, let $A_5 \neq 0$ $\therefore \cos(\lambda l) = 0$

$$\Rightarrow \lambda l = (2n-1)\frac{\pi}{2}\ , n = 1,2,3, \ldots$$

$$\Rightarrow \lambda = \frac{2n-1}{l}\frac{\pi}{2}\ , n = 1,2,3, \ldots$$

Hence from (7) the solutions are

$$u_n(x,t) = a_n \cos\left(\frac{(2n-1)\pi x}{2l}\right) e^{\frac{-K(2n-1)^2\pi^2 t}{4l^2}}, n = 1,2,3, \ldots$$

As equation (1) is linear and homogeneous so by principal of superposition the most general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)\pi x}{2l}\right) e^{\frac{-K(2n-1)^2\pi^2 t}{4l^2}} \qquad (8)$$

Using (4) from (8) we get,

$$u(x,0) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)\pi x}{2l}\right)$$

where

$$a_n = \frac{2}{l}\int_0^l f(x) \cos\left(\frac{(2n-1)\pi x}{2l}\right) dx\ , \qquad (9)$$

$\therefore$ Required solution is given by (8) where a_n is given by (9).

EXAMPLES

1. **Solve the initial boundary-value problem $\frac{\partial^2 u}{\partial x^2} = \frac{1}{K}\frac{\partial u}{\partial t}$ satisfying $u(0,t) = u(l,t) = 0$ and $u(x,0) = lx - x^2$.**

Clearly it is a above type 1 problem. So, the required solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-Kn^2\pi^2 t}{l^2}}$$

Where $B_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\left(\frac{n\pi x}{l}\right) dx$

$$= \frac{2}{l}\left\{\left[-(lx - x^2)\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right)\right]_0^l + \frac{l}{n\pi}\int_0^l (l - 2x)\cos\left(\frac{n\pi x}{l}\right) dx\right\}$$

$$= \frac{2}{l}\left\{0 + \left[\frac{l^2}{n^2\pi^2}(l - 2x)\sin\left(\frac{n\pi x}{l}\right)\right]_0^l - \frac{l^2}{n^2\pi^2}\int_0^l (-2)\sin\left(\frac{n\pi x}{l}\right) dx\right\}$$

$$= \frac{4l}{n^2\pi^2}\int_0^l \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= -\frac{4l^2}{n^3\pi^3}\left[\cos\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= -\frac{4l^2}{n^3\pi^3}[(-1)^n - 1]$$

$$= \frac{4l^2}{n^3\pi^3}[1 - (-1)^n]$$

$$= \begin{cases} 0 & ,when\ n\ is\ even \\ \frac{8l^2}{n^3\pi^3} & ,when\ n\ is\ odd \end{cases}$$

Therefore,

$$u(x,t) = \sum_{n=odd}^{\infty} \frac{8l^2}{n^3\pi^3}\sin\left(\frac{n\pi x}{l}\right) e^{\frac{-Kn^2\pi^2}{l^2}t}$$

$$= \frac{8l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2m-1)^3}\sin\left(\frac{(2m-1)\pi x}{l}\right) e^{\frac{-K(2m-1)^2\pi^2 t}{l^2}}$$

which is the required solution for the given problem.

2. A rod of length 'l' has its ends A and B kept at 0°C and 120°C respectively until steady state condition prevail. If the temperature at B is reduced to 0°C and kept so while that of A is maintained. Find the resulting temperature distribution $u(x,t)$ taking origin at A.

- **When steady state condition prevails:**

The heat equation becomes, $\frac{d^2u}{dx^2} = 0$

$$\Rightarrow u = ax + b$$

where a & b are arbitrary constants.

Given that, $u = 0$ when $x = 0$ and $u = 120$ when $x = l$.

$\Rightarrow a \cdot 0 + b = 0$ and $a \cdot l + b = 120$

$\Rightarrow b = 0$ and $al + b = 120$

Both implying, $a = \frac{120}{l}$, $b = 0$

$\therefore u = \frac{120}{l}x$, which is the solution at steady state condition.

- **After steady state condition:**

The problem is $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$

Subject to, $u(0,t) = u(l,t) = 0$

and $u(x,0) = f(x) = \frac{120}{l}x,\ 0 < x < l$

$\therefore$ The solution is $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-Kn^2\pi^2 t}{l^2}}$

where $B_n = \frac{2}{l}\int_0^l \frac{120}{l} x sin\left(\frac{n\pi x}{l}\right) dx$

$$= \frac{240}{l^2}\left\{\left[-\frac{l}{n\pi} x cos\left(\frac{n\pi x}{l}\right)\right]_0^l + \frac{l}{n\pi}\int_0^l \cos\left(\frac{n\pi x}{l}\right) dx\right\}$$

$$= \frac{240}{l^2}\left\{\frac{l^2}{n\pi}(-1)^{n+1} + \frac{l^2}{n^2\pi^2}\left[\sin\left(\frac{n\pi x}{l}\right)\right]_0^l\right\}$$

$$= \frac{240}{n\pi}(-1)^{n+1}$$

$\therefore$ Required solution is $u(x,t) = \frac{240}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{l}\right) e^{\frac{-Kn^2\pi^2 t}{l^2}}$.

3. Find the temperature distribution u(x,t) of a rod of 20cm whose both ends are immolated and initial temperature is given by f(x) = x(20-x); 0<x<20.

We have to solve $\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2}$

Subject to, boundary conditions $u_x(0,t) = 0 \;\; \forall t \geq 0$

$u_x(20,t) = 0 \;\; \forall t \geq 0$

initial conditions $u(x,0) = f(x) = x(20-x), \;\; 0 < x < 20$

∴ The temperature distribution $u(x,t)$ is given by

$$u(x,t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{20}\right) e^{\frac{-Kn^2\pi^2 t}{400}}$$

where $a_0 = \frac{2}{20}\int_0^{20} x(20-x)dx$

$= \frac{1}{10}\left[10x^2 - \frac{x^3}{3}\right]_0^{20}$

$= \frac{1}{10}400(10 - \frac{20}{3})$

$= \frac{400}{3}$

$a_n = \frac{2}{20}\int_0^{20} x(20-x)\cos\left(\frac{n\pi x}{20}\right)dx$

$= \frac{1}{10}\{\left[\frac{20}{n\pi}x(20-x)\sin\left(\frac{n\pi x}{20}\right)\right]_0^{20} - \frac{20}{n\pi}\int_0^{20}(20-2x)\sin\left(\frac{n\pi x}{20}\right)dx\}$

$= \frac{4}{n\pi}\int_0^{20}(x-10)\sin\left(\frac{n\pi x}{20}\right)dx$

$= \frac{4}{n\pi}\{\left[-\frac{20}{n\pi}(x-10)\cos\left(\frac{n\pi x}{20}\right)\right]_0^{20} + \frac{20}{n\pi}\int_0^{20}\cos\left(\frac{n\pi x}{20}\right)dx\}$

$= \frac{800}{n^2\pi^2}((-1)^{n+1} - 1) + \frac{80}{n^2\pi^2}\frac{20}{n\pi}\left[\sin\left(\frac{n\pi x}{20}\right)\right]_0^{20}$

$= \frac{800}{n^2\pi^2}\{(-1)^{n+1} - 1\}$

$= \begin{cases} 0 & ; n = 2m-1, m = 1,2,3,\ldots \\ -\frac{1600}{n^2\pi^2} & ; n = 2m, m = 1,2,3,\ldots \end{cases}$

$\therefore u(x,t) = \frac{1}{2}\frac{400}{3} + \sum_{n=even}^{\infty}\frac{800}{n^2\pi^2}((-1)^{n+1} - 1)\cos\left(\frac{n\pi x}{20}\right)e^{\frac{-Kn^2\pi^2 t}{400}}$

$= \frac{200}{3} + \sum_{m=1}^{\infty} -\frac{1600}{4m^2\pi^2}\cos\left(\frac{m\pi x}{10}\right)e^{\frac{-Km^2\pi^2 t}{100}}$

$$= \frac{200}{3} - \frac{400}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \cos\left(\frac{m\pi x}{10}\right) e^{\frac{-Km^2\pi^2 t}{100}}$$

which is the required solution.

4. Solve the initial boundary-value problem $\frac{\partial^2 u}{\partial x^2} = K \frac{\partial u}{\partial t}$ satisfying $u_x(0, t) = u(l, t) = 0 \ \forall\, t \geq 0$ and $u(x, 0) = lx - x^2\ , 0 < x < l$.

Clearly it is a above type 4 problem. So, the required solution is

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos\left(\frac{(2n-1)\pi x}{2l}\right) e^{\frac{-K(2n-1)^2\pi^2 t}{4l^2}}$$

Where $a_n = \frac{2}{l}\int_0^l (lx - x^2) \cos\left(\frac{(2n-1)\pi x}{2l}\right) dx$

$$= \frac{2}{l}\left\{\left[(lx - x^2) \cdot \frac{2l}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi x}{2l}\right)\right]_0^l - \int_0^l (l - 2x) \frac{2l}{(2n-1)\pi} \sin\left(\frac{(2n-1)\pi x}{2l}\right) dx\right\}$$

$$= \frac{2}{l} \cdot \frac{2l}{(2n-1)\pi}\left\{0 + \left[(l - 2x) \cdot \frac{2l}{(2n-1)\pi} \cos\left(\frac{(2n-1)\pi x}{2l}\right)\right]_0^l - \int_0^l (-2) \cdot \frac{2l}{(2n-1)\pi} \cos\left(\frac{(2n-1)\pi x}{2l}\right) dx\right\}$$

$$= \frac{4}{(2n-1)\pi}\left\{0 - \frac{2l^2}{(2n-1)\pi} + \frac{4l}{(2n-1)n} \int_0^l \cos\left(\frac{(2n-1)\pi x}{l}\right) dx\right\}$$

$$= \frac{4}{(2n-1)\pi}\left\{- \frac{2l^2}{(2n-1)\pi} + \frac{4l}{(2n-1)n}\left[\frac{2l}{(2n-1)\pi} \cdot \sin\left(\frac{(2n-1)\pi x}{2l}\right)\right]_0^l\right\}$$

$$= \frac{4}{(2n-1)\pi}\left\{- \frac{2l^2}{(2n-1)\pi} + \frac{8l^2}{(2n-1)n} \cdot (-1)^n\right\}$$

$$= -\frac{8l^2}{(2n-1)^2\pi^2} + \frac{32l^2}{(2n-1)^3\pi^3} \cdot (-1)^n$$

$$= \frac{8l^2}{(2n-1)^2\pi^2}\left[\frac{4}{(2n-1)\pi}(-1)^n - 1\right]$$

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8l^2}{(2n-1)^2\pi^2}\left[\frac{4}{(2n-1)\pi}(-1)^n - 1\right] \cos\left(\frac{(2n-1)\pi x}{2l}\right) e^{\frac{-K(2n-1)^2\pi^2 t}{4l^2}}$$

CONCLUSION

This is just a brief note about Heat Equation where we have mostly discussed about homogeneous heat equation, its derivations and solutions. We also discussed how various factors like boundary conditions and some examples on how to solve heat equation & studied some applications of heat equation in science. From the above discussion we can say that PDEs such as heat equation is one of the crucial PDEs to solve and study about heat equation in various conditions.

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BANKURA CHRISTIAN COLLEGE

BANKURA ,2022-23

WAVE EQUATION

A PROJECT REPORT

SUBMITTED BY

L A K S M I K A N T A M A N D A L

UID:21013121004

REGISTRATION NO.:00331 OF 2021-22

SUBJECT: MATHEMATICS

COURSE TITLE: PROJECT WORK

COURSE ID: 62117

COURSE CODE: SH/MTH/604/DSE-4

EXAMINATION: B.Sc. SEMISTER -VI HONOURS EXAMINATION,2022-23

CERTIFICATE

*This is to certify that **LAKSHMIKANTA MANDAL (UID: 21013121004, Registration No.: 00331 of 2021-22) of DEPARTMENT OF MATHEMATICS, BANKURA CHRISTIAN COLLEGE, BANKURA,** has successfully carried out this project work entitled "WAVE EQUATION" under my supervision and guidance.*

This project has been undertaken as a part of the Curriculum of Bankura University (Semester – VI, Paper: DSE-4 (Project Work)) (Course ID: 62117) and for the partial fulfillment for the degree of Bachelor of Science (Honours) in Mathematics of Bankura University in 2023 – 24.

(Signature of the Supervisor) *(Signature of Head of the Department)*

Department of Mathematics, Bankura Christian College

Department of Mathematics

Bankura Christian College

Submitted for UG Semester – VI Project Viva-voce Examination, 2022 –23 in Mathematics (Honours) (Paper: DSE-4 (Project Work)) held on ________________.

Signature of Internal Examiner(s) *Signature of External Examiner*

Date: Date:

DECLARATION

I hereby declare that my project, titled 'Wave Equation', submitted by me to Bankura University for the purpose of DSE-4 paper, in semester VI under the guidance of my professor of Mathematics in Bankura Christian College,Dr.Subhasis Bandyopadhyay.I also declare that the project has not been submittedhere by any other student.

Name: ***LAKSHMIKANTA MANDAL***

UID Number: ***21 01 31 21 004***

Semester: ***VI***

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INDEX

INTRODUCTION

"We live in a world of waves" said Stewart. Mathematical description of wave phenomena is one of the fundamentals not only in mechanics but also in many other areas of physics. Water waves, sound waves, seismic waves, electromagnetic waves (radio waves, light waves, X-Rays, Gamma rays), etc. are known and studied intensively because they are around us and have a wide range of applications in real life. For several such reasons wave equation is acknowledged as one of the 17 equations that changed the world. It is the hyperbolic type. It typically concerns a time variable t, one or more spatial variables, $x_1, x_2, \dots, x_n$ *and a scalar function* $u = u(x_1, x_2, \dots, x_n; t)$ *whose values could model, for example, the mechanical displacement of a wave. The hyperbolic equations are connected with initial-boundary conditions and pure initial conditions. This equation play very important role in the study of applied mathematics and physics*

The history of the wave equation is related to names such as Jean d'Alembert, Leonhard Euler, Daniel Bernoulli, Luigi Lagrange and Joseph Fourier. The debate on proper solution of the wave equation between d'Alembert, Euler and Bernoulli during the 18th century has formulated the basics of the analysis and gave impetus to further studies.

The wave equation describes the propagation of an excitation generated by initial or boundary condition with a constant speed c. It is a hyperbolic partial differential equation of *second degree. It typically concerns a time variable t, one or more spatial variables* $x_1, x_2, \ldots, x_n$ *and a scalar function* $u = u(x_1, x_2, \ldots., x_n; t)$ *whose values could model, for example, the mechanical displacement of a wave.*

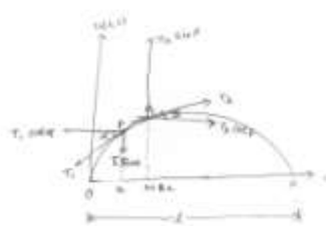
The wave equation for u is

$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$ *,Where* ∇^2 *is the (spatial) and c is a fixed constant.*

Derivation of Wave Equations

The two cases we will consider are waves traveling along a string under tension, and geometryof wave. The methods of derivation are rather different, and they illustrate two of the main approaches for obtaining governing equations in many other situations .

I. Transverse waves in a string under tension



Let a string of length l be attached at two end O and A. Let u(x,t) be a displacement at any distance x and anytime t.

Let P and Q two neighbouring point on the string corresponding to the distancex and $x + \delta x$ *respectively.*

We obtain the equation of motion of the string under following assumptions:

1) The string is perfectly flexible and there is no resistance.

$\Longrightarrow$*The tension of the string is tangential to the car at each point.*

Let the tension at P and Q are T_1 *and* T_2 *respectively. If inclination of tension* T_1 *and* T_2 *with horizontal be* α *and* β *then*

$$tan\,\alpha \;= \frac{\partial u}{\partial x}\,|_P \;and\; tan\;\beta = \frac{\partial u}{\partial x}|_Q$$

2) String moves only vertical direction and there is no motion along horizontal direction. Therefore, the sum of the force in horizontal direction must be zero

i.e. $T_2\;cos\,\beta - (T_1\,cos\,\alpha) = 0$

$\Longrightarrow T_1\,cos\,\alpha = T_2\;cos\,\beta = T = constant$

3) Gravitational force on the string is neglected .

By Newton second law motion, mass × acceleration = sum of forces

$$m\delta x\;\frac{\partial^2 u}{\partial t^2} = T_2\,sin\,\beta - T_1\;sin\,\alpha$$

$$\Longrightarrow \frac{m\delta x}{T}\;\frac{\partial^2 u}{\partial t^2} = \frac{T_2\,sin\,\beta}{T} - \frac{T_1\;sin\,\alpha}{T}$$

$$\Longrightarrow \frac{\partial^2 u}{\partial t^2} = \frac{1}{\frac{m\delta x}{T}}\,[T_2\,sin\,\beta - T_1\;sin\,\alpha]$$

$$\Longrightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T_2\,sin\,\beta}{T_2\;cos\,\beta} - \frac{T_1\;sin\,\alpha}{T_1\;cos\,\alpha} \quad [since\;T_1\,cos\,\alpha = T_2\;cos\,\beta = T]$$

$\Longrightarrow \frac{\partial^2 u}{\partial t^2} = \frac{T}{m}\frac{1}{\delta x}\left[\tan\alpha \ - \tan\ \beta\right]$

$= \frac{T}{m}\frac{1}{\delta x}\left[\frac{\partial u}{\partial x}\ |_P - \frac{\partial u}{\partial x}|_Q\right]$

$= \frac{T}{m}\ \left[\frac{u_x(x+\delta x, t) - u_x(x,t)}{\delta x}\right]$

$= \frac{T}{m}\ u_{xx}(x,t)\ \ as\ \delta x \longrightarrow 0$

$\Longrightarrow \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

Which is an one dimensional wave equation of a string.

II. DERIVATION USING GEOMETRY OF WAVE

Wave can be expressed in the form $f(x \pm ct)$

By geometry propagating waves: When the wave propagates as result of time t increasing ,it maintains its shape. The propagated wave is a copy of the original wave shifted to the right, or left,by the distance ct, c is the wave propagation speed.

Consider $f(x - ct)$ *and consider small changes in* x *and* t*, i.e.*Δx,Δt. *They each cause a small shift or translation of* $f(x - ct)$

Note that $\Delta x = c\Delta t$ *(since c is the propagation speed of the wave)*

So, $\frac{\Delta f}{\Delta x} = \frac{\Delta f}{c\Delta t} = \frac{1}{c}\frac{\Delta f}{\Delta t}$

Then repeating that we get $\frac{\Delta^2 f}{\Delta^2 x} = \left(\frac{1}{c}\right)^2 \frac{\Delta^2 f}{\Delta^2 t}$

Letting Δ become a very small we get $\frac{\partial^2 f}{\partial x^2} = \left(\frac{1}{c}\right)^2 \frac{\partial^2 f}{\partial t^2}$

From the geometry alone, it was only needed to note that a change in t multiplied by the velocity yields the same results (as measured by the second derivative) as a change in x that is ,a translation of $f(x - ct)$.

- *1-D wave equation in its standard form is* $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

  Example of 1-D waves are: waves on a string /spring.

- *2-D wave equation in its standard form is* $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

  Example of 2-D waves are: water waves

- *3-D wave equation in its standard form is* $\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$

  Example of 3-d waves are: light and sound waves.

SOLUTION OF 1-D WAVE EQUATION

SOLUTION OF ONE DIMENSIONAL WAVE EQUATION BY CANONICAL REDUCTION

The one dimensional way equation is given by

$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$ ……………………………………………..(1)

Equality wave equation (1) with

$A \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial t} + C \frac{\partial^2 u}{\partial t^2} = 0$

Here, A= -c^2,B = 0 & C= 1

Discreminant D = B² - 4AC

$$= 0 - 4(-c^2)$$

$$= 4c^2 > 0$$

It follows that the one dimensional wave equation

$$A\lambda^2 + B\lambda + C = 0$$

$$or, \lambda^2 - C^2 = 0$$

$$\lambda = \pm C$$

The characteristic equation corresponding to wave equation are $\frac{dx}{dt} \pm c = 0$

The solution corresponding to ODE are

$x + ct = c_1$ *and* $x - ct = c_2$ *where* c_1*and* c_2 *are constant*

Let us choose $\xi = x + ct$ *and* $\eta = x - ct$

$\xi_x = 1, \ \xi_t = c, \xi_{xt} = \xi_{tx} = 0 \ , \xi_{xx} = \xi_{tt} = 0$

$\eta_x = 1, \ \eta = c, \eta_{xt} = \eta_{tx} = 0 \ , \eta_{xx} = \eta = 0$

$J = \begin{vmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{vmatrix} = \begin{vmatrix} 1 & c \\ 1 & -c \end{vmatrix} = -2c \neq 0$

Therefore, $a(\xi, \eta) = A\xi_x^2 + B\xi_x\xi_t + C\xi_t^2$

$$= c^2 - c^2$$

$$= 0$$

$b(\xi, \eta) = \frac{D}{A} = -4c^2$

$c(\xi, \eta) = A\eta_x^2 + B\eta_x\eta_t + C\eta_t^2$

$$= c^2 - c^2 = 0$$

Thus the wave question (1) reduce as to

$au_{\xi\xi} + bu_{\xi\eta} + cu_{\eta\eta} = f(\xi, \eta)$

Or, $bu_{\xi\eta} = f(\xi,\eta)$

Or, $u_{\xi\eta} = 0$*,Which is the canonical form of wave equation*

Integrating with respect to ξ

$u_\eta = \alpha(\eta)$ *Where is a function of alone*

Integrating with respect to η

$$u = \int \alpha(\eta)\, d\eta + \varphi(\xi)$$

$$= \psi(\eta) + \varphi(\xi)$$

$$= \varphi(c + ct) + \psi(x + ct)$$

Which is general solution of one dimensional wave equation .

I. *FOR INFINITE STRING :*

The corresponding wave equation will be $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ ………………(1) $-\infty < x < \infty$,

with initial condition $u(x,0) = f(x), u_t(x,0) = g(x)$, *that is we specify the initial position and initial velocity of the string .*

(This is known as Cauchy problem)

Let us introduce new independent variables:

$$\xi = x + ct\ \eta = x - ct\ ,$$

So that

$$x = \frac{\xi + \eta}{2}\ t = \frac{\xi - \eta}{2c}$$

Then the new function is $w(\xi,\eta) = u(x,y) = u(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c})$

To obtain a differential equation for ω *, we differentiate with respect to* ξ:

$$w_{\xi} = \frac{1}{2}u_x\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right) + \frac{1}{2c}u_t\left(\frac{\xi+\eta}{2},\frac{\xi-\eta}{2c}\right)$$

and then differentiate with respect to η :

$$w_{\xi\eta} = \frac{1}{4}u_{xx} - \frac{1}{4c}u_{xt} + \frac{1}{4c}u_{tx} - \frac{1}{4c^2}u_{tt}$$

Since $u_{xt} = u_{tx}$ *, the two middle term cancel ,and we obtain from (1) that*

$w_{\xi\eta} = 0$ *...........(2)*

Now w_{ξ} *must be independent of* η *, so* $w_{\xi} = \varphi_1(\xi)$*, for some function* φ_1*, and integrating this with respect to* ξ *, we obtain that* $w(\xi,\eta) = \varphi(\xi) + \psi(\eta)$ *, for some function* φ *, such that* $\varphi = \varphi_1$*, and some function* ψ *, the "constant of integration" . Returning to our original variables we obtain*

$u(x,t) = \varphi(x+ct) + \psi(x-ct)$ *...............................(3)*

Now such function u with arbitrary differentiable functions φ *and* ψ *satisfies equation (1). So obtain a general solution which depends on two arbitrary functions.*

Equation (1) describes oscillations of an infinite string or a wave in 1-dimensionalmedium.Tosingle out a unique solution we impose initial condition at $t = 0$*:*

$u(x,0) = f(x)u_t(x,0) = g(x)$*.......................................(4)*

To simplify our computation , we can use the ***Superposition Principle***

First we find a solution with arbitrary given f and g =0, then we find a solution with $f = 0$ *and arbitrary g, and taking the sum of these two solutions .*

For the first solution we plug $t = 0$ *into (3) and obtain*

$$\varphi + \psi = f \ \varphi + \psi = 0 \,,$$

So $\varphi = \psi = \frac{f}{2}$*, and the first solution ,with zero initial velocity , is*

$u_1(x,t) = \frac{1}{2}(f(x+ct) + g(x-ct))$ ……………………………………….(5)

For the second solution we differentiate (3) with resect to t and plug $t = 0$*.*

We obtain

$$\varphi + \psi = 0 \,, c(\varphi' - \psi') = g$$

Solving this we obtain the second solution

$u_2(x,t) = \frac{1}{2c}\int_{x-ct}^{0} g(y)dy + \frac{1}{2c}\int_{0}^{x+ct} g(y)dy$ ………………………………….(6)

Corresponding to zero initial position.

Thus the complete solution u of the initial value problem (1) , (4) is given by

$$u(x,t) = u_1(x,t) + u_2(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$$

This is the ***d'Alembert formula.***

<u>NOTE</u>

Cauchy problem for the one dimensional non-homogeneous wave question $\boldsymbol{\frac{\partial^2 u}{\partial t^2}} -$

$\boldsymbol{c^2 \frac{\partial^2 u}{\partial x^2} = F(x,t)}, -\infty < x < \infty, t > 0$

Subject to the initial conditions

$\boldsymbol{u(x,0) = f(x), u_t(x,0) = g(x), 0 \leq x < \infty}$

Sol.

$$\boldsymbol{u(x,t) = u_1(x,t) + u_2(x,t)}$$

$$\boldsymbol{= \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy}$$

$$\boldsymbol{+ \frac{1}{2c}\int_{\alpha=0}^{t}\int_{\beta=x-c(t-\alpha)}^{x+c(t-\alpha)} F(\alpha,\beta)\, d\alpha d\beta}$$

II. FOR SEMI-INFINITE STRING :

♦ ***With fixed End***

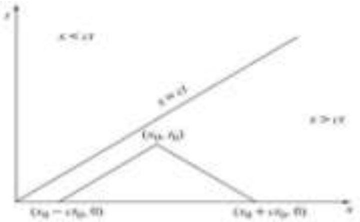
Let us first consider a semi-infinite vibrating string with a fixed end, that is,

be$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, 0 < x < \infty, t > 0$

$$u(x,0) = f(x), u_t(x,0) = g(x), 0 \leq x < \infty$$

$u(0,t) = 0, 0 \leq t < \infty$ *. [this is known as Dirichlet Boundary Condition]*

It is evident here that the boundary condition at $x = 0$ *produces a wave moving to the right with the velocity c. Thus, for* $x > ct$*, the solution is the same as that ofthe infinite string, and the displacement is influenced only by the initial data on theinterval* $[x - ct, x + ct]$.



When $x < ct$, *the interval [x − ct, x + ct] extends onto the negative x-axis wherefand g are not prescribed. But from the d'Alembert formula*

$$u(x,t) = \varphi(x+ct) + \psi(x-ct),$$

Where

$$\varphi(\xi) = \frac{1}{2}\,f(\xi) + \frac{1}{2c}\int_0^{\xi} g(y)dy + \frac{k}{2}$$

$$\psi(\eta) = \frac{1}{2}\,f(\eta) + \frac{1}{2c}\int_0^{\eta} g(y)dy - \frac{k}{2}$$

We see that $u(0,t) = \varphi(ct) + \psi(-ct) = 0.$

Hence , $\psi(-ct) = -\varphi(ct).$

If we let $\alpha = -ct$*, then* $\psi(\alpha) = -\varphi(-\alpha)$

Replacing α *by* $x - ct$*, we obtain for x<ct,*

$$\psi(x-ct) = -\varphi(ct-x),$$

$$\text{and hence },\psi(x-ct) = \frac{1}{2}f(ct-x) - \frac{1}{2c}\int_0^{ct-x} g(y)dy - \frac{k}{2}$$

The solution of the initial boundary value problem , therefore is given by

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy \qquad \textit{for } x > ct,$$

$$u(x,t) = \frac{1}{2}\big(f(x+ct) - f(ct-x)\big) + \frac{1}{2c}\int_{ct-x}^{x+ct} g(y)dy \qquad \textit{for } x < ct\ ,$$

♦ ***With free End***

We consider a semi-infinite string with a free end at $x = 0$.

We will determine the solution of

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}\ , 0 < x < \infty, t > 0$$

$$u(x,o) = f(x), u_t(x,0) = g(x), 0 \leq x < \infty,$$

$u_x(0,t) = 0, 0 \leq t < \infty$. *[This is known as Neumann's Boundary Condition]*

As in the case of the fixed end ,for $x > ct$ *the solution is the same as that of the infinite string .*

For $x < ct$, *from the d'Alembert solution*

$$u(x,t) = \varphi(x+ct) + \psi(x-ct),$$

We have , $u_x(x,t) = \varphi^{'}(x+ct) + \psi^{'}(x+ct)$.

Thus, $u_x(0,t) = \varphi^{'}(ct) + \psi^{'}(-ct) = 0$

Integration yields $\varphi(ct) - \psi(-ct) = k$, *where k is a constant .*

Now , if we let $\alpha = -ct$ *, we obtain* $\psi(\alpha) = \varphi(-\alpha) - k$

Replacing α *by* $x - ct$ *, we have* $\psi(x-ct) = \varphi(ct-x) - k$

And hence , $\psi(x-ct) = \frac{1}{2}\ f(ct-x) + \frac{1}{2c}\int_0^{ct-x} g(y)dy\ - \frac{k}{2}$

The solotion of the initial boundary-value problem , therefore ,is given by

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy \qquad for\ x > ct$$

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(ct-x)] + \frac{1}{2c}\left[\int_0^{x+ct} g(y)dy + \int_0^{ct-x} g(y)dy\right] for$$

$x < ct.$

III. FOR FINITE STRING WITH FIXED END :

We first consider the vibration of the string of length l fixed at both ends.

The problem is that of finding the solution of

We will determine the solution of

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial t^2},\ 0 < x < l, t > 0 \quad \text{...(1)}$$

$u(x,0) = f(x),\ u_t(x,0) = g(x),\ 0 \le x < l\,,$

$$u(0,t) = 0\,, \qquad u(l,t) = 0\,, \quad t \ge 0$$

From the previous results, we know that the solution of the wave equation is

$$u(x,t) = \varphi(x+ct) + \psi(x-ct).$$

Applying the initial condition,we have

$$u(x,0) = \varphi(x) + \psi(x) = f(x), 0 \le x < l$$

$u_t(x,0) = c\varphi'(x) - c\psi'(x) = g(x)\,, \quad 0 \le x < l$

Solving for φ and ψ, we find

$$\varphi(\xi) = \frac{1}{2}\ f(\xi) + \frac{1}{2c}\int_0^{\xi} g(y)dy\ + \frac{k}{2}, 0 \le \xi \le l, \quad \text{.......................................(2)}$$

$$\psi(\eta) = \frac{1}{2}\ f(\eta) + \frac{1}{2c}\int_0^{\eta} g(y)dy - \frac{k}{2}\ ,0 \le \eta < l \quad \text{.......................................(3)}$$

Hence ,

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct}\ g(y)dy \quad \text{......................(4)}$$

For $0 \le x + ct < l$, *and* $0 \le x - ct < l$

The solution is thus uniquely determined by the initial data in the region

$$t \leq \frac{x}{c}, t \leq \frac{l-x}{c}, \quad t \geq 0$$

For large times, the solution depend on the boundary conditions. Applying the boundary conditions , we obtain

$u(0,t) = \varphi(ct) + \psi(-ct) = 0\,, \qquad t \geq 0$ ………………….(5)

$u(l,t) = \varphi(l+ct) + \psi(l-ct) = 0\,, t \geq 0$ ……………………(6)

If we set $\alpha = -ct$ *, equation (5) becomes*

$\psi(\alpha) = -\varphi(-\alpha), \alpha \leq 0$ ……………………………………………….(7)

And if we set $\alpha = l + ct$ *, equation (6) takes the form*

$\varphi(\alpha) = -\psi(2l-\alpha), \quad \alpha \geq l$ ……………………………………………(8)

With $\xi = -\eta$*, we may write the equation* (2) *as*

$\varphi(-\eta) = -\frac{1}{2}f(\eta) + \frac{1}{2c}\int_0^{-\eta} g(y)dy + \frac{k}{2}, \; 0 \leq -\eta \leq l$ ………(9)

Thus , from (7) and (9), we have

$\psi(\eta) = -\frac{1}{2}f(-\eta) - \frac{1}{2c}\int_0^{2l-\xi} g(y)dy - \frac{k}{2}, \; -l \leq \eta \leq 0$ ………..(10)

We see that the range of $\psi(\eta)$ *is extended to* $-l \leq \eta \leq l$ *. If we put* $\alpha = \xi$ *in the equation (8) ,*

We obtain $\varphi(\xi) = -\psi(2l-\xi)\,, \; \xi \geq l$ ………………………………(11)

Then, by putting $\eta = 2l - \xi$ *in the equation (3), we obtain*

$\psi(2l-\xi) = \frac{1}{2}f(2l-\xi) - \frac{1}{2c}\int_0^{2l-\xi} g(y)dy - \frac{k}{2}\;, \; 0 \leq 2l-\xi \leq l$ ………..(12)

Substitution of this , in the equation (11) yields

$\varphi(\xi) = -\frac{1}{2}f(2l-\xi) + \frac{1}{2c}\int_0^{2l-\xi} g(y)dy + \frac{k}{2}\,, l \leq \xi \leq 2l$ ……………(13)

The range of $\varphi(\xi)$ *is thus extended to* $0 \le \xi \le 2l$ *. Continuing in this manner , we obtain* $\varphi(\xi)$ *for all* $\xi \ge 0$ *and* $\psi(\eta)$ *for all* $\eta \le l$ *.*

Hence , the solution is determind for all $0 \le x \le l$ *and* $t \ge 0$ *.*

EXAMPLES

1. ***Let u(x,t) be the solution of the following equation*** $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $x \in \mathbb{R}$, $t > 0$,

   Subject to the initial condition $u(x, 0) = f(x) = 1 \;\; if \; |x| \le 1$

   $= 0 \;\; if \; |x| > 1$

   and $u_t(x, 0) = g(x) = 1 \;\; if \; |x| \le 1$

   $= o \;\; if \; |x| > 1$

   Find $u\left(0, \frac{1}{4}\right)$

$\Longrightarrow$ *We know that d'Alembert formula for Cauchy problem is given by*

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$$

Here c=2 ,

Therefore, $u(x,t) = \frac{1}{2}[f(x+2t) + f(x-2t)] + \frac{1}{2.2}\int_{x-2t}^{x+2t} g(y)dy$

Hence, $u\left(0, \frac{1}{4}\right) = \frac{1}{2}\left[f\left(\frac{1}{2}\right) + f\left(-\frac{1}{2}\right)\right] + \frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} g(y)dy$

$$= \frac{1}{2}[1+1] + \frac{1}{4}\int_{-\frac{1}{2}}^{\frac{1}{2}} 1.\, dy$$

$$= 1 + \frac{1}{4}\left[\frac{1}{2} — \frac{1}{2}\right)$$

$= 1 + \frac{1}{4} = \frac{5}{4}$

2. ***We know that d'Alembert formula for Cauchy problem is given by***

   Find the solution of the initial value problem be $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, $x \in \mathbb{R}$, $t > 0$, $u(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{\sin x}$, $\boldsymbol{u_t}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{\cos x}$.

⇒Clearly it is an infinite string problem , and the solution obtained by the d'Alembert formula $u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$

Here we have $u(x,0) = \sin x$, $u_t(x,0) = \cos x$. *Then we have* $f(x) = \sin x$ *and* $g(x) = \cos x$. *Then*

$$u(x,t) = \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos y \, dy$$

$$= \sin x \cos ct + \frac{1}{2c}[\sin(x+ct) - \sin(x-ct)]$$

$$= \sin x \cos ct + \frac{1}{c}\cos x \sin ct \, .$$

3. ***Solve the initial value problem***

$$\boldsymbol{u_{tt}} = \boldsymbol{u_{xx}} \, , -\infty < x < \infty \, , t > 0$$

$$\boldsymbol{u}(\boldsymbol{x}, \boldsymbol{0}) = \boldsymbol{e}^{-x^2} \, , \boldsymbol{x} \in \mathbb{R}$$

$$\boldsymbol{u_t}(\boldsymbol{x}, \boldsymbol{t}) = \boldsymbol{0} \, ,$$

⇒ *Here* , $f(x) = e^{-x^2}$ $\quad g(x) = 0 \quad c = 1$

By d'Alembert formula

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy$$

$$= \frac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right]$$

4. Determine the solution of the initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, x > 0, t > 0,$$

$$u(x,0) = |\sin x|, x > 0$$

$$u_t(x,0) = 0, x \geq 0$$

$$u(x,0) = 0, t \geq 0.$$

⇒*Clearly it is a problem of semi-infinite string with free end and the nsolution is given by*

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy \qquad for\ x > ct,$$

$$u(x,t) = \frac{1}{2}\big(f(x+ct) - f(ct-x)\big) + \frac{1}{2c}\int_{ct-x}^{x+ct} g(y)dy \qquad for\ x < ct$$

Here $c^2 = 4 f(x) = |\sin x|, x > 0 \qquad and\ g(x) = 0$

For $x > 2t$,

$$u(x,t) = \frac{1}{2}\big(f(x+2t) + f(x-2t)\big)$$

$$= \frac{1}{2}[|\sin(x+2t)| - |\sin(x-2t)|]$$

And for $x < 2t$,

$$u(x,t) = \frac{1}{2}[f(x+2t) - f(2t-x)] = \frac{1}{2}[|sin(x+2t)| - |sin(2t-x)|.$$

We notice that $u(0,t) = 0$ *is satisfied by* $u(x,t)$ *for* $x < 2t$ *(that is* $t > 0$*).*

5. ***Determine the solution of the initial boundary value problem***

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \ , \mathbf{0} < x < \infty \ , t > 0,$$

$$u(x,0) = cos\left(\frac{\pi x}{2}\right) , 0 \le x < \infty$$

$$u_t(x,0) = 0 \ , 0 \le x < \infty$$

$$u_x(x,0) = 0 \ , t > 0$$

$\Rightarrow$ *Clearly it is a problem of semi-infinite string with fixed ends , and the solution is given by*

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy \qquad for\ x > ct$$

$$u(x,t) = \frac{1}{2}\big(f(x+ct) + f(ct-x)\big) + \frac{1}{2c}\left[\int_0^{x+ct} g(y)dy + \int_0^{ct-x} g(y)dy\right], for\ x < ct$$

Here $c^2 = 1 f(x) = cos\left(\frac{\pi x}{2}\right) \qquad g(x) = 0$

For $x > t$,

$$u(x,t) = \frac{1}{2}[f(x+t) + f(t-x)]$$

$$= \frac{1}{2}\left[cos\left(\frac{\pi}{2}(x+t)\right) + cos\left(\frac{\pi}{2}(t-x)\right)\right]$$

$$= cos\frac{\pi x}{2} cos\frac{\pi t}{2}$$

APPICATION OF WAVE EQUATION

Seismology

Seismology is the scientific study of earthquakes and the propagation of elastic waves through the Earth. It involves analyzing seismic waves generated by earthquakes or other sources (like explosions) to understand the Earth's internal structure, composition, and processes.

The wave equation is fundamental in seismology as it describes how seismic waves propagate through the Earth. The basic form of the wave equation for seismic waves is:

$$v^2\nabla^2 u - \frac{\partial^2 u}{\partial t^2} = 0$$

where: u represents the displacement vector of the wave,

∇^2 *is the Laplace operator (representing spatial derivatives),* $\frac{\partial^2 u}{\partial t^2}$ *is the second derivative of u with respect to time, v is the speed of the seismic wave.*

This equation governs how seismic waves (such as P-waves, S-waves, and surface waves) travel through different materials and structures within the Earth. In practice, seismologists use this equation, along with data from seismic stations around the world, to *determine properties such as earthquake locations, magnitudes, and depths, as well as to map the Earth's interior.*

The wave equation allows seismologists to interpret the arrival times, amplitudes, and frequencies of seismic waves recorded by seismometers to infer details about the Earth's subsurface. By studying how these waves propagate through different geological materials, seismologists can create detailed models of the Earth's crust, mantle, and core.

Oceanography

Oceanography is the scientific discipline that studies the ocean and its various aspects, including its physical, chemical, biological, and geological properties. It involves understanding ocean currents, waves, tides, marine life, coastal processes, and the interactions between the ocean and the atmosphere.

One of the fundamental tools in studying waves in oceanography is the wave equation. The wave equation describes the propagation of waves through a medium, such as water in the ocean.

This equation relates the acceleration of a small element of water (surface elevation) to the gravitational force acting on it and its position. Oceanographers use this equation (and more complex forms depending on the situation) to model and predict the behavior of waves in different oceanic conditions.

Modeling Waves: *Oceanographers use mathematical models based on the wave equation to simulate how waves behave under different conditions, such as wind speed, water depth, and bottom topography.*

Predicting Wave Behavior: *By solving the wave equation, oceanographers can predict characteristics of waves like their height, wavelength, and propagation direction.*

Understanding Ocean Dynamics: *Waves play a crucial role in ocean circulation, mixing of water masses, coastal erosion, and other processes. Understanding wave behavior helps in understanding these broader oceanographic phenomena.*

Engineering and Design: *Engineers use wave equations to design structures such as offshore platforms, coastal defenses, and ships that can withstand the forces exerted by waves.*

In summary, the wave equation is a powerful tool in oceanography that helps scientists understand and predict the behavior of waves in the ocean, contributing to broader studies of ocean dynamics and their impact on marine environments and human activities.

IMPORTANCE

IN MATHEMATICS:

Under partial differential equation, wave equation is used to calculate the displacementof one-dimensional wave. The wave equation alone does not specify a physical solution;a unique solution is usually obtained by setting a problem with further conditions, suchas initial conditions, which prescribe the amplitude and phase of the wave. Using thedesired PDE under different initial and boundary conditions various real-life problemcould be solved. Another important class of problems occurs in enclosed spacesspecified by boundary conditions, for which the solutions represent standing waves, orharmonics, analogous to the harmonics of musical instruments.

IN PHYSICS:

The classical wave equation is a cornerstone in mathematical physics and mechanics. Itsmodifications are widely used in order to describe wave phenomena.

Electromagnetic Theory:*Maxwell's equations, which describe electromagnetism, can be combined to form wave equations that describe how electric and magnetic fieldspropagate as electromagnetic waves. This is crucial for the theory of light and the entirefield of optics.*

Relativity and Gravitational Waves:*In the context of general relativity, the wave equation helps describe the propagation of gravitational waves, ripplesin spacetime caused by accelerating masses. This is a significant area of research inastrophysics and cosmology.*

Description of Wave Propagation:*The wave equation provides a mathematical framework for describing how waves propagate through different mediums. This includes mechanical waves in solids (like seismic waves), electromagnetic waves (such as light and radio waves), and acoustic waves (like sound waves).*

Prediction and Analysis:*By solving the wave equation, physicists and engineers can predict the behavior of waves in different scenarios. This is crucial for designing devices that use waves (e.g., antennas, ultrasound machines) and understanding wave interactions with boundaries and obstacles.*

Computational Physics: *Numerical solutions of the wave equation are critical forsimulations in computational physics. These simulations are used in weatherprediction, seismic analysis, and many other scientific and engineering applications.*

In summary, the wave equation is a cornerstone of both theoretical and applied physics, providing critical insights and tools for understanding and manipulating wavephenomena across different fields and scales.

IN CHEMISTRY:

For Schrodinger equation for quantum wave functions, governing set of PDEs produceabsolutely accurate result.The wave equation, specifically the Schrödinger waveequation, is fundamental in chemistry for several reasons:

Foundation for Quantum Mechanics:In quantum mechanics, the wave equationforms the basis for the Schrödinger equation, which describes how the quantum stateof a physical system changes over time. This equation is essential for understanding thebehavior of particles at the quantum level.*

Reaction Mechanisms: *Understanding the electronic structure of molecules helps inpredicting and explaining chemical reactions and mechanisms. It helps chemistsunderstand how and why reactions occur, the energy changes involved, and thetransition states.*

Spectroscopy:The solutions to the Schrödinger equation are used to predict theenergy levels of electrons in atoms and molecules. These energy levels correspond tothe absorption and emission spectra, which are used in various spectroscopictechniques to analyze and identify substances.*

Understanding Chemical Bonding:Quantum mechanics, based on the wave equation, explains how atoms bond to form molecules. It provides insights into the types of chemical bonds that can form (ionic, covalent, metallic) and how these bonds influence the structure, stability, and properties of molecules.*

Overall, the Schrödinger wave equation provides a comprehensive framework forunderstanding the microscopic properties of matter, which are essential forexplainingmacroscopic chemical phenomena.

CONCLUSION

Here are the key points to remember from these Notes.

First, there are many solutions to the wave equation. It is only by adding boundary and initial conditions that we turn a question about the wave equation into a well-posed mathematical problem.

Second, for the constant coefficient wave equation, there are many techniques for computing the solutions to (well-posed) problems. These included derivation of wave equation, d'Alembert formula , Cauchy problem, examples of wave equation. A technical facility with these techniques will help you in solving real problems about waves. Finaly, for non-constant coefficient problems, we still expect many solutions to exist for the wave equation. Well-posedness will come from imposing boundary and initial conditions.

Local solutions can always be found. We can optimistically expect that the many solutions to the constant coefficient case can be applied, at least locally, to give methods of solution to the more general, non-constant coefficient case.

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CAUCHY EULER ODE

A PROJECT REPORT

SUBMITTED BY

SANCHITA KARMAKAR

FOR

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BANKURA CHRISTIAN COLLEGE BANKURA

2023-24

DEPARTMENT OF MATHEMATICS **MATHEMATICS(HONOURS)**

BANKURA CHRISTIAN COLLEGE **BANKURA UNIVERSITY**

CAUCHY EULER ODE

A PROJECT REPORT

UNDER THE SUPERVISION OF Dr. SUBHASIS BANDYOPADHYAY

SUBMITTED BY

SANCHITA KARMAKAR

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BANKURA CHRISTIAN COLLEGE

BANKURA

2023-24

DECLARATION

I hereby declare that my project, titled **"Cauchy Euler ODE"** , was submitted by me to **Bankura University** for a **DSE-4 paper**, in **semester VI** under the guidance of my professor of Mathematics at **Bankura Christian College**, **Dr. Subhasis Bandyopadhyay**.

I also declare that the project has not been submitted here by any other student.

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Semester**: VI**

ACKNOWLEDGEMENT

First of all, I am immensely indebted to my institution, Bankura Christian College under Bankura University. It is my humble pleasure to acknowledge my deep gratitude to our principal sir, Dr. Fatik Baran Mandal, for the valuable suggestions and encouragement that made this project successful.

I am grateful to the HOD of our Mathematics department, Dr. UtpaL Kumar Samanta, for always helping in every situation. A heartful thanks to my guide in this project, Dr. Subhasis Bandyopadhyay, whose guidance and valuable support have been instrumental in completing this project work.

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INTRODUCTION

It has been seen that in case of an n-th order linear differential equation with a constant coefficient, the form of the complementary function may be readily determined. The general n-th order linear equation with variable coefficients is quite a different matter , however, and only in certain special cases can the complementary function be obtained explicity in closed form. One special case of considerable practical importance for which it is fortunate that this can be done is the so-called Cauchy-Euler equation or Equidimensional equation. This equation is the most important, if not the only higher order linear differential with variable coeeficients encountered in typical ordinary differential equations courses. A passing knowledge of history should give us pause. No matter how precocious Augustin-Louis Cauchy(1789-1857) was, he and Leonard Euler (1707-1783) never collaborated, even under the generous Hardy-Littlewood guidelines. This does not necessarily mean that it is unfounded to put their names together. There are still obvious reasons these mathematicians could be connected to this equation and each other. Maybe Euler proposed the problem and Cauchy solved it. Perhaps Euler proved a simple case and Cauchy placed it in final form. Possibly they arrived at solutions independently. George Simmons provides a hint in his textbook [20, p.86]. He defines this equation as the Euler equidimensional equation with the footnote. Euler's researchers were so extensive that many mathematicians try to avoid confusion by naming equations, formulas, theorems, etc., for the person who first studied them after Euler. In this contemporary world, not only the practitioners but also researchers use this to solve various problems of ordinary differential equations that have been coming into the calculation part.

<u>IMPORTANCE</u>

The second order Cauchy-Euler equations are used in various fields of science and engineering such as in time-harmonic vibrations of a thin elastic rod, problems on annual and solid disc , wave mechanics ,etc.

The applications of Euler's formula extend far into both theoretical and practical areas. Knowing these real-world applications appreciate the power and elegance of this formula. In **<u>Electrical Engineering</u>**, the formula is indispensable in simplifying calculations that involve alternating current circuits, where complex numbers are used to represent quantities like voltage and current. This allows mathematicians and engineers to transform real-time differential equations into algebraic equations. **<u>Signal Processing</u>** is another field heavily drawing from Euler's formula. Fourier transfom, a pivotal signal processing algorithm, relies on Euler's formula to transform signals between time and frequency domains. This lets you understand the frequency components of a signal , beneficial in image processing, audio processing and telecommunications. Euler's formula and its ilk made appearances in **<u>quantum mechanics</u>**. Quantum states, particularly wave functions, can indeed be expressed using Euler's formula , aiding the understanding of atom behaviour and the nature of light.

DERIVATION OF CAUCHY – EULER ODE

The differential equation

$$a_n x_n y(n) + a_{n-1} x_{n-1} y(n-1) + \cdots + a_0 y = 0$$

is called the Cauchy-Euler differential equation of order n. the symbols a_i, i = 0,…,n are constants and an not equal to zero. the Cauchy-Euler equation is important in the theory of linear differential equations because it has direct application to Fourier's method in the study of partial differential equations. in particular, the second-order Cauchy-Euler equation

$$ax^2y'' + bxy' + cy = 0$$

accounts for almost all such applications in applied literature. a second argument for studying the Cauchy-Euler equation is theoretical: it is a single example of a differential equation with non- constant coefficients that has a known closed-form solution. this fact is due to a change in variables. $(x, y) \rightarrow (t, z)$given by equations $x=e^t$ and z(t) = y(x), which changes the Cauchy-Euler equation into a constant-coefficient differential equation. since the constant coefficient equations have closed-form solutions, so also do the Cauchy-Euler equations.

GENERAL SOLUTION OF CAUCHY – EULER ODE:-

1.Second order – solving through trial solution

The most common Cauchy–Euler equation is the second-order equation, appearing in several physics and engineering applications, such as when solving Laplace's equation in polar coordinates. The second-order Cauchy-Euler equation is:

$$x^2\frac{d^2y}{dx^2}+ax\frac{dy}{dx}+by=0$$

We assume a trial solution: $y = x^m$. Differentiating gives:

$$\frac{dy}{dx}=mx^{m-2}$$

And,

$$\frac{d^2y}{dx^2}=m(m-1)x^{m-2}$$

Substituting into the original equation leads to requiring

$$x^2(m(m-1)x^{m-2})+ax(mx^{m-1})+b(x^m)=0$$

Rearranging and factoring gives the indical equation

$$m^2+(a-1)m+b=0.$$

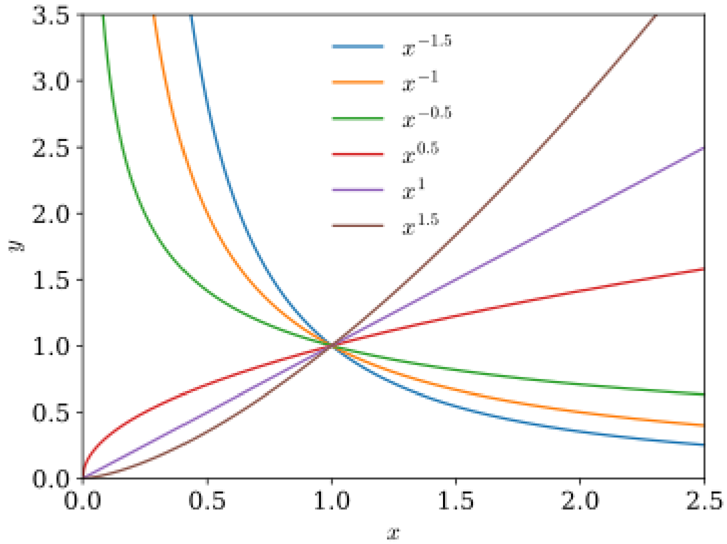
We then solve for m. There are three particular cases of interest :

- Case 1 of two distinct roots, m_1 and m_2;
- Case 2 of one real repeated root, m;
- Case 3 of complex roots, $\alpha \pm \beta i$.

In case 1, the solution is

$$y = C_1 x^{m_1} + C_2 x^{m_2}$$

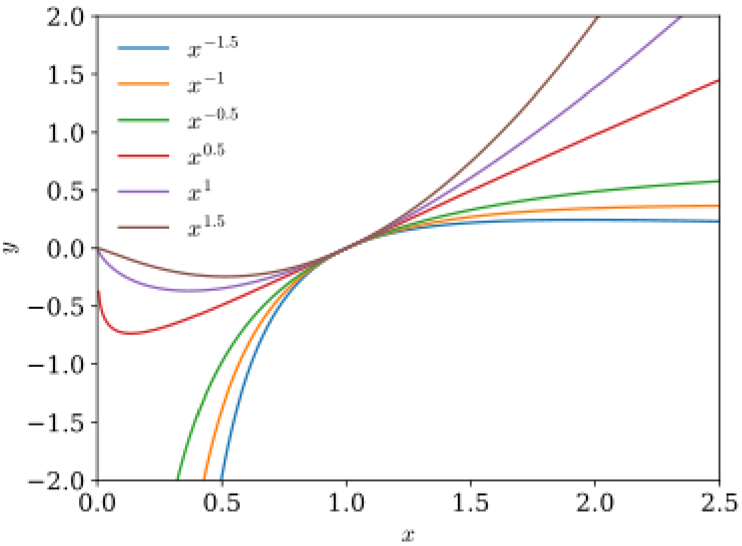
Typical solution curves for a second-order Euler–Cauchy equation for the case of two real roots



In case 2, the solution is

$$y = C_1 x^m \ln(x) + C_2 x^m$$

Typical solution curves for a second-order Euler–Cauchy equation for the case of a double root



To get to this solution , the method of reduction of order must be applied after having found one solution $y = x^m$.

In case 3, the solution is:

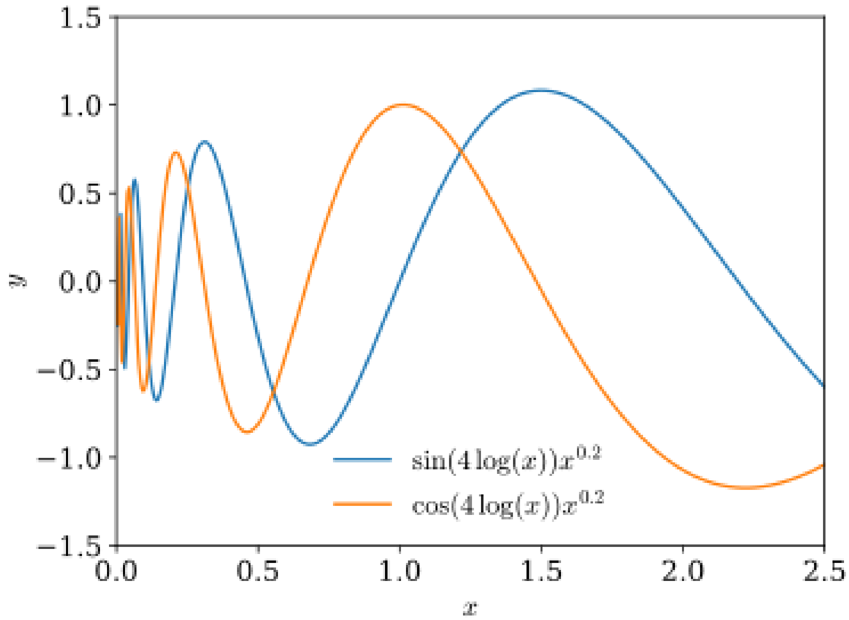
$$y = C_1 x^{\alpha} \cos(\beta \ln(x)) + C_2 x^{\alpha} \sin(\beta \ln(x))$$

$$\alpha = \text{Re}(m)$$

$$\beta = \text{Im}(m)$$

This form of the solution is derived by setting $x = e^t$ *and using* Euler's formula.

Typical solution curves for a second-order Euler–Cauchy equation for the case of complex roots



2.Second order - solution through change of variable

$$x^2 \frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$$

We operate the variable substitution defined by,

$$t = \ln(x)$$

$$y(x) = \varphi(\ln(x)) = \varphi(t).$$

Differentiating gives,

$$\frac{dy}{dx} = \frac{1}{x}\frac{d\varphi}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}(\frac{d^2\varphi}{dt^2} - \frac{d\varphi}{dt}).$$

Substituting $\varphi(t)$, the differential equation becomes:

$$\frac{d^2\varphi}{dt^2} + (a-1)\frac{d\varphi}{dt} + b\varphi = 0.$$

This equation in $\varphi(t)$ is solved via its characteristic polynomial:

$$\lambda^2 + (a-1)\lambda + b = 0.$$

Now let λ1 and λ2 denote the two roots of this polynomial. We analyze the case where there are distinct roots and the case where there is a repeated root:

If the roots are distinct, the general solution is:

$$\varphi(t) \;= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},$$

Where the exponentials may be complex.

If the roots are equal, the general solution is

$$\varphi(t) \;= c_1 e^{\lambda_1 t} + c_2 t e^{\lambda_1 t}.$$

In both cases, the solution, y(x) may be found by setting $t = ln(x)$

Hence, in the first case,

$$y(x) = C_1 x^{\lambda_1} + C_2 x^{\lambda_2},$$

And in the second case, $y(x) = C_1 x^{\lambda_1} + C_2\, ln(x)\, x^{\lambda_1}$.

3.Second order - solution using differential operators

Observe that we can write the second order Cauchy-Euler equations in terms of a linear differential operator L as

$$Ly = (x^2D^2 + axD + bI)y = 0,$$

Where $D = \frac{d}{dx}$ and I is the identity operator.

We express the above operator as a polynomial in **xD** rather than **D**. By the product rule,

$$(xD)^2 = xD(xD) = x(D + xD^2) = x^2D^2 + xD.$$

So,

$$L = (xD)^2 + (a-1)(xD) + bI.$$

We can then use the quadratic formula to factor this operator into linear terms. More specifically, let λ1 and λ_2 denote the (possibly equal) values of

$-\frac{a-1}{2} \pm \frac{1}{2}\sqrt{(a-1)^2 - 4b}$.

Then,

$$L = (xD - \lambda_1 I)(xD - \lambda_2 I).$$

These factors commute,that is, $(xD - \lambda_1 I)(xD - \lambda_2 I) = (xD - \lambda_2 I)(xD - \lambda_1 I)$. Hence, if $\lambda_1 \neq \lambda_2$, the solution to $Ly = 0$, is a linear combination of the solutions to each of $(xD - \lambda_1 I)y = 0$ and $(xD - \lambda_2 I)\, y = 0$, which can be solved by separation of variables. Indeed, with $i \in \{1,2\}$.

We have $(xD - \lambda_i I)y = x\frac{dy}{dx} - \lambda_i y = 0$.

So,

$$x\frac{dy}{dx} = \lambda_i y$$

$$\int \frac{1}{y} dy = \lambda_i \int \frac{1}{x} dx$$

$$ln\, y = \lambda_i\, ln\, x + C$$

$$y = C_i e^{\lambda_i \ln x} = C_i x^{\lambda_i}.$$

Thus , the general solution is $y(x) = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$. If $\lambda = \lambda_1 = \lambda_2$, then we instead need to consider the solution of $(xD - \lambda I)^2 y = 0$. Let $z = (xD - \lambda I)$

y, so that we can write $(xD - \lambda I)^2 y = (xD - \lambda I)\, z = 0$. As before, the solution of $(xD - \lambda I)\, z = 0$ is of the form $z = C_1 x^{\lambda}$. So, we are left to solve

$(xD - \lambda I)y = x\frac{dy}{dx} - \lambda y = C_1 x^{\lambda}$. We then rewrite the equation as:

$$\frac{dy}{dx} - \frac{\lambda}{x} y = C_1 x^{\lambda - 1},$$

Which one can recognize as being amenable to solution via an integrating factor.

Choose $M(x) = x^{-\lambda}$ as our integrating factor . Multiplying our equation through by $M(x)$ and recognizing the left hand side as the derivative of a product, we then obtain:

$$\frac{d}{dx}\left(x^{-\lambda} y\right) = C_1 x^{-1}$$

$$x^{-\lambda} y = \int C_1 x^{-1}\, dx$$

$$y = x^{\lambda}(C_1\, ln(x) + C_2)$$

$$= C_1 x^{-1}\, C_1\, ln(x) x^{\lambda} + C_2 x^{\lambda}.$$

THEOREM

THEOREM: The transformation $x = e^t$ reduces the equation

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} x \frac{dy}{dx} + a_n y = F(x) \text{--------(1)}$$

to a linear differential equation with constant coefficients.

Proof: This is what we need! We shall prove this theorem for the case of second-order Cauchy-Euler differential equation

$$a_0 x^2 \frac{d^2 y}{dx^2} + a_1 x \frac{dy}{dx} + a_2\, y = F(x) \text{--------(2)}$$

The proof in the general nth-order case proceeds in a similar fashion. Letting $x = e^t$, assuming $x > 0$, we have $t = ln(x)$. Then

$$\frac{dy}{dx} = \frac{\mathrm{dy}}{\mathrm{dt}}\frac{\mathrm{dt}}{\mathrm{dx}} = \frac{1}{x}\frac{dy}{dt}$$

And

$$\frac{d^2 y}{dx^2} = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{1}{x}\left(\frac{d^2 y}{dt^2}\frac{dt}{dx}\right) - \frac{1}{x^2}\frac{dy}{dt}$$

$$\frac{1}{x}\left(\frac{d^2 y}{dt^2}\frac{1}{x}\right) - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right)$$

Thus

$$x\frac{dy}{dx} = \frac{dy}{dt} \text{ and } x^2\frac{d^2 y}{dx^2} = \frac{d^2 y}{dt^2} - \frac{dy}{dt}$$

Substituting into equation (2) we obtain

$$a_0\left(\frac{d^2 y}{dt^2} - \frac{dy}{dt}\right) + a_1\frac{dy}{dt} + a_2\, y = F(e^t)$$

Or

$$A_0\frac{d^2 y}{dt^2} + A_1\frac{dy}{dt} + A_2\, y = G(t) \text{--------(3)}$$

Where $A_0 = a_0,\ A_1 = a_1 - a_0, A_2 = a_2, G(t) = F(e^t).$

This is a second-order Cauchy-Euler differential equation with constant coefficients, which was we wished to show.

EXAMPLES

1.Solve a Cauchy-Euler equation step by step. Consider the second-order Cauchy-Euler equation:

x^2y′′−6xy′+13y=0

SOLUTION:

Assume a solution of the form $y = x^r$.

The derivatives are $y' = rx^{r-1}$ *and* $y'' = r(r-1)x^{r-2}$.

Substitute y, y', and y'' back into the original equation:

$$x^2\,(r(r-1)x^{r-2}) - 6x(\,rx^{r-1}) + 13x^r = 0$$

Factor out x^r since $(x \neq 0)$ and solve the characteristic equation:

$$r(r-1) - 6\,r + 13 = 0$$

$$\Rightarrow r^2 - 7r + 13 = 0$$

This is a quadratic equation in r.

Since the discriminant $b^2 - 4ac$ is negative , the roots of the characteristic equation are complex.

Let's find the roots:

$$r = \frac{7 \pm \sqrt{49-4(1)(13)}}{2}$$

$$\Rightarrow \quad r = \frac{7 \pm \sqrt{49-52}}{2}$$

$$\Rightarrow \quad r = \frac{7 \pm \sqrt{-3}}{2}$$

$$\Rightarrow \quad r = \frac{7}{2} \pm \frac{\sqrt{3}i}{2}$$

The roots are complex hence, the general solution is:

$$y(x) = x^{7/2}(C_1 \cos\left(\frac{\sqrt{3}}{2}\ln(x)\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}\ln(\varkappa)\right))$$

Here, C_1 and C_2 are constants determined by boundary conditions or initial values.

2.Solve: x^2y′′−xy′+5y=0

SOLUTION:

The characteristic equation takes the form

$r(r-1)-r+5=0$

or

$r^2-2r+5=0$

The roots of this equation are complex, $r_{1,2}=1\pm 2i$.

Therefore, the general solution is $y(x)=x(c_1\cos(2\ln|x|)+c_2\sin(2\ln|x|))$.

3.Solve the initial value problem: t2y″+3ty′+y=0 , With the initial conditions y(1)=0,y′(1)=1.

SOLUTION:

For this example the characteristic equation takes the form

$r(r-1)+3r+1=0$

or

$r^2+2r+1=0$

There is only one real root, $r=-1$.

Therefore, the general solution is

$y(t)=(c_1+c_2\ln|t|)t^{-1}$

However, this problem is an initial value problem.

At $t=1$ we know the values of y and y'.

Using the general solution, we first have that

$$0 = y(1) = c_1$$

Thus, we have so far that $y(t) = c_2 \ln|t|\, t^{-1}$.

Now, using the second condition and

$$y'(t) = c_2\,(1 - \ln|t|)\,t^{-2}$$

we have

$$1 = y(1) = c_2$$

Therefore, the solution of the initial value problem is $y(t) = \ln|t|\, t^{-1}$.

CONCLUSION

The learners have to change Cauchy-Euler equation into a constant coefficient differential formula. Hence, these constant coefficient equations consist of closed form solutions that can be solved through Cauchy's second degree equation. In summary, the Cauchy-Euler differential equation can be solved by transforming it into a characteristic equation. The nature of the roots of this characteristic equation (whether the are distinct real, repeated real, or complex) determines the form of the general solution. This method provides a structured approach to find solutions to this class of differential equations.

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LAPLACE TRANSFORM

A PROJECT REPORT

SUBMITTED BY

LOPAMUDRA SAINI

UID - 21013121032

FOR

B.Sc. Semester – VI
Examination 2023-24

IN

MATHEMATICS (HONOURS)

BANKURA UNIVERSITY

DEPARTMENT OF MATHEMATICS

BANKURA CHRISTIAN COLLEGE

LAPLACE TRANSFORM

A PROJECT REPORT

Under the Supervision of

Dr. Subhasis Bandyopadhyay

SUBMITTED BY

LOPAMUDRA SAINI

UID - 21013121032

REGISTRATION NO. – 00359 of 2021-22

SUBJECT – MATHEMATICS

COURSE TITLE – PROJECT WORK

COURSE ID – 62117

COURSE CODE – SH/MTH/604/DSE-4

EXAMINATION – B.Sc. Semester –VI Honours Examination, 2023-24

CERTIFICATE

This is to certify that Lopamudra Saini (UID: 21013121032, Registration No: 00359) of Department of Mathematics, Bankura Christian College, Bankura, has successfully carried out this project work entitled "LAPLACE TRANSFORM" under my supervision and guidance.

This project has been undertaken as a part of the Curriculum of Bankura University (Semester - VI, Paper: DSE-4 (Project Work)) (Course ID: 62117) and for the partial fulfillment for the degree of Bachelor of Science (Honours) in Mathematics of Bankura University in 2023 - 24.

_________________________ _________________________________

Signature of Supervisor

Signature of Head of the Department
(Dept. of Mathematics,
Bankura Christian College)

Name of the Supervisor:
(Department of Mathematics,
Bankura Christian College)

Submitted for UG Semester - VI Project Viva-voce Examination, 2023-24 in Mathematics (Honours) (Paper: DSE-4 (Project Work)) held on

_______.

_____________________________ ______________________________

Signature of Internal Examiner(s)
Date :

Signature of the External Examiner
Date :

DECLARATION

I hereby declare that my project, titled "Laplace Transform" submitted by me to Bankura University for the purpose of DSE-4 paper, in semester VI under the guidance of my professor of Mathematics in Bankura Christian College, Dr. Subhasis Bandyopadhyay.

I also declare that the project has not been submitted here by any other student.

Name: Lopamudra Saini

UID Number: 21013121032

Semester : VI

ACKNOWLEDGEMENT

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CONTENTS

INTRODUCTION

An operator takes a function as input and outputs another function. A Transform does the same thing with the added twist that the output function has a different independent variable. i.e. A transform is a mathematical operation that converts a function or a signal from one domain to another domain. There are many types of transforms, among them in this project we will discuss about *Laplace Transform*, a powerful mathematical tool used to solve differential equations, analyze control systems, process signals and many more.

Laplace transform is named in honour of the great French mathematician, *Pierre Simon De Laplace* (1749-1827), who used a similar transform (now called z transform) in his work on probability theory. The current widespread use of the transform came about soon after World War II although it had been used in the 19th century by Abel, Lerch, Heaviside and Bromwich

Like all transforms, the Laplace transform changes one signal into another according to some fixed set of rules or equations. The Laplace transform is a mathematical operation that converts a function of real variable t (Usually in the time domain) i.e. f(t), into a function of a complex variable *s* (Usually in the complex - valued frequency domain) i.e. F(s).

The Laplace transform is defined as the integral:

$$\mathrm{F(s)} = \int_0^\infty f(t)\, e^{-st}\, dt$$

The best way to convert differential equations into algebraic equations is the use of Laplace transformation. In this project, we embark on a journey to explore the intricacies of Laplace transform and delve into its myriad applications. Through theoretical insights, practical examples, and real-world case studies, we aim to unravel the mysteries behind this transformative mathematical tool and showcase its indispensable role in modern science and technology.

1.LAPLACE TRANSFORM:

1.1 Definition:

The Laplace transform is a mathematical operation that converts a function of real variable t (Usually in the time domain) i.e. f(t), into a function of a complex variable s (Usually in the complex - valued frequency domain) i.e. F(s).

The Laplace transform is defined as the integral:

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

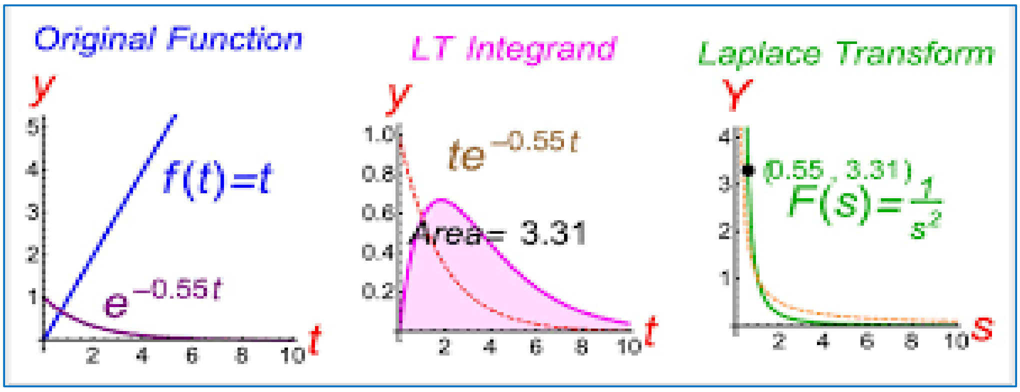
1.2. Example:

Find the Laplace transform of the function f(t)=1

Solution: $L\{1\} = \int_0^\infty e^{-st}\, dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \left[\frac{e^{-\infty}-e^{0}}{-s}\right] = \frac{1}{s}$$



Laplace transform does not exist for all functions. If it exists, it is uniquely determined. The following conditions are to be satisfied:

Let $\int_0^\infty f(t)\, e^{-st}\, dt$ exists form s > a, if

1. f(t) is peicewise continuous on every finite interval

2. $f(t)$ Satisfies the following equality: $|f(t)| \leq b.e^{at}$ for all t≥0 and for some constants a and b.

Then L [f(t)] exists.

1.1. Some basic Laplace transform :

f (t)	t^n	e^{at}	sinat	cosat	sinhat	coshat	e^{at}sinbt	e^{at}cosbt
F(s)	$\frac{n!}{s^{n+1}}$	$\frac{1}{s-a}$	$\frac{a}{s^2+a^2}$ s > 0	$\frac{s}{s^2+a^2}$ s > 0	$\frac{a}{s^2-a^2}$	$\frac{s}{s^2-a^2}$	$\frac{b}{(s+a)^2+b^2}$	$\frac{s-a}{(s+a)^2+b^2}$

2.INVERSE LAPLACE TRANSFORM:

2.1. Definition:

The inverse Laplace Transform is a mathematical operation that reverses the process of taking Laplace Transforms. It converts a function from the Laplace domain, where complex numbers are used, (i.e. F (s)) back to the original time domain (i.e. f (t)).

And it is denoted by : L^{-1} { F (s) } = f (t)

2.2. Example: Find the Laplace transform of the function f(t) = $\frac{1}{s}$

Solution: $L^{-1} \{ \frac{1}{s} \} = 1$

2.3. Some basic Inverse Laplace transform :

F(s)	$\frac{1}{s^{n+1}}$	$\frac{1}{s-a}$	$\frac{a}{s^2+a^2}$ s > 0	$\frac{s}{s^2+a^2}$ s > 0	$\frac{a}{s^2-a^2}$	$\frac{s}{s^2-a^2}$
f (t)	$\frac{t^n}{n!}$	e^{at}	sinat	cosat	sinhat	coshat

3.PROPERTIES OF LAPLACE TRANSFORM:

3.1. Linearity:

If $f_1(t), f_2(t)$ be any function of t and a_1, a_2 are ant constant, then :

$$L\{a_1f_1(t) + a_2f_2(t)\} = a_1L\{f_1(t)\} + a_2L\{f_2(t)\}$$

3.2. First Shifting Property :

If F(s) is the Laplace transform of the function f(t) then F(s-a) is the Laplace transform of $e^{at}f(t)$. Where a is real or complex numbers

i.e. $L\{f(t)\} = F(s)$ then $L\{e^{at} f(t)\} = F(s-a)$

Example: Find the Laplace transform of the function $t^4 e^{3t}$

Solution: Let $F(t) = t^4$, then

$$L\{t^4\} = \frac{4!}{s^5} = \frac{24}{s^5} = F(s)$$

Now, $L\{ e^{3t} t^4 \} = F(s-3)$

$$= \frac{24}{(s-3)^5}$$

3.2.1.First Shifting Property for inverse Laplace transform:

If $L^{-1}\{ F(s) \} = f(t)$, then for any constant(real or complex) a

$$L^{-1}\{ F(s-a) \} = e^{at} f(t) = e^{at} L^{-1}\{ F(s) \}$$

Example: Find the Laplace transform of $t^4 e^{3t}$

Solution: Let $f(t) = t^4$, then

$$L^{-1}\left\{ \frac{24}{(s-3)^5} \right\} = e^{3t} L^{-1}\left\{ \frac{24}{s^5} \right\}$$

$$= e^{3t} t^4$$

A function of type $\begin{cases} 1, & t \geq a \\ 0, & t \leq a \end{cases}$ is called unit step function.

It is denote by H $(t-1)$

3.3. Second Shifting Property :

Let f(t) be a function of t, & L {f(t)} = F (s) then

L{ f(t-a) H (t-a) } = e^{-as} F(s)

Example: Find the Laplace transform of $\begin{cases} t-1 & t \geq 1 \\ 0 & t < 1 \end{cases}$

Solution: We can write it as (t −1) $\begin{cases} 1 & t \geq 1 \\ 0 & t < 1 \end{cases}$

$$L (t) = (t -1) H (t -1) = \frac{1}{s^2}$$

Now L { (t −1) H (t −1) } = $e^{-s} \frac{1}{s^2}$

3.3.1.Second Shifting Property for inverse Laplace transform :

Let f(s) be a inverse Laplace transform of f(t) such that L^{-1}\{ F(s)\} = f(t) then

$$L^{-1}\{ e^{-as} F (s) \} = f (t-a) H (t-a),$$

Where H(t-a) is a unit step function.

Example: Find $L^{-1}\left\{ \frac{e^{-\frac{\pi}{3}s}}{s^2+1} \right\}$

Solution: Here F (s) = $\frac{1}{s^2+1}$ then, L^{-1} { F(s) } = sin t

So, $L^{-1}\left\{ \frac{e^{-\frac{\pi}{3}s}}{s^2+1} \right\} = \sin\left(t - \frac{\pi}{3} \right) H \left(t - \frac{\pi}{3} \right)$

$$= \begin{cases} \sin\left(t - \frac{\pi}{3}\right) & t \geq \frac{\pi}{3} \\ 0 & t < \frac{\pi}{3} \end{cases}$$

3.4. Convolution Property for inverse Laplace transform :

3.4.1. Convolution of two function :

If F(t) and G(t) are two functions, then the convolution of F and G is denoted by F*G and is defined by

$$F*G = \int_0^t F(u)\, G\,(t-u)du$$

Example: Find $L^{-1}\{ \frac{1}{(S-1)(S+2)} \}$

Solution: Let F (s) = $\frac{1}{(S-1)}$ & G(s) = $\frac{1}{(S+2)}$

Then, $L^{-1}\{ f(s) * g(s)\} = \int_0^t F(u)\, G\,(t-u)du$

$$= \int_0^t e^{u} e^{-2(t-u)} du$$

$$= e^{-2t} \int_0^t e^{3u} du$$

$$= e^{-2t}$$

3.5. Change of scale property :

If L { f(t) } = F(s), then L { f (at) } = $\frac{1}{a}$ F ($\frac{S}{a}$)

Example: If L { f $\sqrt{t}$ } =1/ s $\sqrt{}$1+s^2 , Find L { f (2$\sqrt{t}$) }

Solution: L { f $\sqrt{4t}$ } = 1/4 [1 / ($\frac{S}{4}$ $\sqrt{}$(1+($\frac{S}{4}$)^2))]

= 1 / (s $\sqrt{}$(1+ s^2/16))

= 4 / (s $\sqrt{}$(s^2+16))

Therefore, L { f (2$\sqrt{t}$) } = 4 / (s $\sqrt{}$(s^2+16))

3.6. Multiplicative Property:

Let f(t) be a function and if L { f (t) } = F (s) then

L { t f(t) } = - d/ds F(s) & L { t^n f(t) } = $(-1)^n$ d^n / ds^n F (s)

Example: Let g(t) = t cos t, then find L { g(t) }

Solution: Let f(t) = cos t

Then L{ f(t) } = s / s^2 + 1

Now L{ t f(t) } = - d/ds { s / s^2 + 1 }

= - (- s^2 + 1 / $(s^2 + 1)^2$)

= s^2 + 1 / $(s^2 + 1)^2$

Therefore, L { g(t) } = L { t cost } = s^2 + 1 / $(s^2 + 1)^2$

3.6.1. Multiplicative Property for inverse Laplace transform :

Let f(s) be a Laplace transform of a function f(t) then

L^{-1} { F (s) } = f (t) and L^{-1} { F `(s) } = -t f(t) ⇨ $L^{-1}$ { F `(s) } = -t L^{-1} { F (s) }

Example: Find L^{-1} { $\log\left(\frac{s-4}{s+3}\right)$ }

Solution: Here F (s) = $\log\left(\frac{s-4}{s+3}\right)$ = log (s-4) – log (s-3)

Now, F`(s) = $\frac{1}{s-4} - \frac{1}{s+3}$

L^{-1} { F `(s) } = $L^{-1}\left\{\frac{1}{s-4}\right\}$ - $L^{-1}\left\{\frac{1}{s+3}\right\}$

⇨ L^{-1} { F `(s) } = e^{4t} - e^{-3t}

⇨ -t L^{-1} { F (s) } = e^{4t} - e^{-3t}

⇨ L^{-1} { F(s) } = $\frac{-(e^{4t} - e^{-3t})}{t} = \frac{e^{-3t} - e^{4t}}{t}$

Therefore L^{-1} { $\log\left(\frac{s-4}{s+3}\right)$ } = $\frac{e^{-3t} - e^{4t}}{t}$

3.7. Division Property :

If f(s) be a Laplace transform of f(t) i.e. L{ f(t) } = F(s), then

$$L \{ f(t) / t \} = \int_0^\infty F(s)\ ds$$

Example: Find the Laplace transform of sin t / t

Solution: We know that L (sin t) = 1 / s^2 + 1

Then, L { sin t / t } = $\int_0^\infty$ 1 / s^2 + 1 ds

= [$\tan^{-1} s]_0^\infty$

= $\pi / 2 - \tan^{-1} s$

= $\cot t^{-1}$ s

NOTE: L { cos t / t } does not exist as log ∞ does not exist.

3.7.1. Division Property for inverse Laplace transform :

Let L^{-1} { F (s) } = f(t), then

$$L^{-1} \{ F(s) / s \} = \int_0^\infty f(t)\ dt$$

Example: Find inverse Laplace transform of 1 / s (s^2 +4)

Solution: We know that L^{-1} { 1 / (s^2 +4) } = sin 2t / 2

L^{-1}\{ 1 / s (s^2 +4) \} = 1/2 $\int_0^\infty \sin 2t\, dt$

= -1/2 [cos 2t /2 $]_0^t$

= -1/4 [cos2t – 1]

= 1/4 [1-cos2t]

3. THEOREMS OF LAPLACE TRANSFORM

1. Initial Value Theorem :

The initial value theorem of Laplace transform enables us to calculate the initial value of a function x(t) [i.e.,x(0)] directly from its Laplace transform X(s) without the need for finding the inverse Laplace transform of X(s).

Statement :

The initial value theorem of Laplace transform states that, if

$$x(t) \overset{LT}{\leftrightarrow} X(s)$$

Then, $\lim_{t\to0} x(t) = x(0) = \lim_{s\to\infty} s\,X(s)$

Proof :

From the definition of unilateral Laplace transform, we have,

$L[x(t)] = X(s) \int_0^\infty x(t)\, e^{-st}\, dt$

Taking differentiation on both sides, we get,

$$L\left[\frac{dx(t)}{dt}\right] = \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt$$

By the time differentiation property [i.e. $\frac{dx(t)}{dt} \overset{LT}{\leftrightarrow} sX(s) - x(0^-)$]

of Laplace transform, we get,

$L\left[\frac{dx(t)}{dt}\right] = \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt = sX(s) - x(0^-)$

Now, taking $\lim_{s\to\infty}$ on both sides, we have,

$\lim_{s\to\infty} \left\{ \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt \right\} = \lim_{s\to\infty} \{sX(s) - x(0)\}$

$\Rightarrow 0 = \lim_{s\to\infty} sX(s) - x(0)$

$\Rightarrow x(0) = \lim_{s\to\infty} sX(s)$

Therefore, we have,

$$\lim_{t\to0} x(t) = x(0) = \lim_{s\to\infty} sX(s)$$

2. Final Value Theorem :

The final value theorem of Laplace transform enables us to find the final value of a function x(t) [i.e. x(∞)] directly from its Laplace transform X(s) without the need for finding the inverse Laplace transform of X(s).

Statement :

The final value theorem of Laplace transform states that, if

$x(t) \overset{LT}{\leftrightarrow} X(s)$

Then, $\lim_{t\to\infty} x(t) = x(\infty) = \lim_{s\to 0} s\,X(s)$

Proof :

From the definition of the unilateral Laplace transform, we have

$L[x(t)] = X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt$

Taking differentiation on both sides, we get,

$L\left[\frac{dx(t)}{dt}\right] = \int_0^{\infty} \frac{dx(t)}{dt}\, e^{-st}\, dt$

By the time differentiation property [i.e. $\frac{dx(t)}{dt} \overset{LT}{\leftrightarrow} sX(s) - x(0^-)$]

of Laplace transform, we get,

$L\left[\frac{dx(t)}{dt}\right] = \int_0^{\infty} \frac{dx(t)}{dt}\, e^{-st}\, dt = sX(s) - x(0^-)$

Now, taking $\lim_{s\to 0}$ on both sides, we have,

$\lim_{s\to 0} \left\{ \int_0^{\infty} \frac{dx(t)}{dt}\, e^{-st}\, dt \right\} = \lim_{s\to 0} \{sX(s) - x(0-)\}$

$\Rightarrow \int_0^{\infty} \frac{dx(t)}{dt}\, dt = \lim_{s\to 0} \{ sX(s) - x(0^-) \}$

$\Rightarrow x(\infty) - x(0^-) = \lim_{s\to 0} sX(s)$

$\Rightarrow x(\infty) = \lim_{s\to 0} sX(s)$

Therefore, we have,

$$\lim_{t\to\infty} x(t) = x(\infty) = \lim_{s\to 0} s\,X(s)$$

3. Convolution Theorem :

If f(s) and g(s) are Laplace transform of F(t) and G(t) respectively

i.e. If $L^{-1}\{ f(s) \} = F(t)$ and $L^{-1}\{ g(s) \} = G(t)$ then

$L^{-1}\{ f(s) * g(s)\} = \int_0^t F(u)\, G\,(t-u)du$

or

$L^{-1}\{ f(s) * g(s)\} = \int_0^t G(u)\, F\,(t-u)du$

4. SOLVING ODE BY LAPLACE TRANSFORM

Let f (y′) = Q be an ordinary differential equation,
then take Laplace transform both sides

$L\{f\,(y')\} = L(Q)$

Where, $L\{y'\} = sL(y) - y(0)$
$L(y'') = s^2L(y) - sy(0) - y'(0)$

Problem: Solve the initial value problem $y''+ 4y=0$, $y(0) =1$, $y'(0) =3$

Solution: Transforming the equation, we have

$$0 = s^2 y - sy\,(0) - y'(0) + 4$$

Solving for Y , we have

$$Y(s) = \frac{s+3}{s^2+4}$$

Splitting the expression into two terms, we have

$$Y(s) = \frac{s}{s^2+4} + \frac{3}{s^2+4}$$

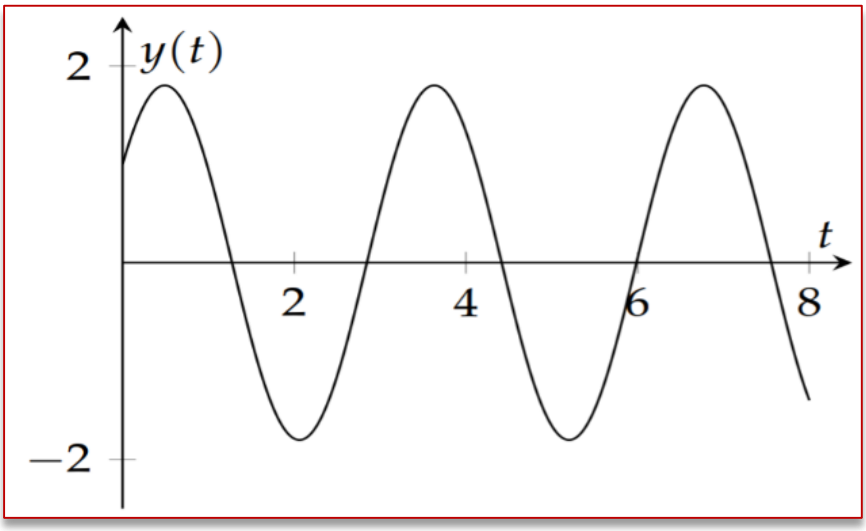
The first term is now recognizable as the transform of cos2t.The second term is not the transform of sin2t. It would be if the numerator were a 2 .

This can be corrected by multiplying and dividing by 2

$$\frac{3}{s^2+4} = \frac{3}{2}\left(\frac{2}{s^2+4}\right)$$

The solution is then found as

$$y(t) = L^{-1}\left[\frac{s}{s^2+4} + \frac{3}{2}\left(\frac{2}{s^2+4}\right)\right] = \cos 2t + \frac{3}{2}\sin 2t$$



5. SOLVING PDE BY LAPLACE TRANSFORM

Problem: Solve: $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$

BC: $u(x, 0) = 6e^{-3x}$, $x > 0, t > 0$

Solution: Taking Laplace transform of both sides of the given PDE,

$$L \left\{ \frac{\partial u}{\partial x}\right\} = 2L \left\{ \frac{\partial u}{\partial t} \right\} + L\{u\}$$

$$\frac{dU}{dx} = 2[\, sU - u\,(x,0)\,] + U$$

By the given Boundary Conditions [BC].

$$\frac{dU}{dx} = -\,(2s+1)U = -12e^{-3x}$$

Which is an ordinary linear differential equation in U

Its Integrating Factor (I.F.) = $e^{-\int(2s+1)dx} = e^{-(2s+1)x}$

Therefore the solution is

$$U\, e^{-(2s+1)x} = C - 12\int e^{-3x}\, e^{-(2s+1)x}\, dx$$

$$= C - 12\int e^{-(2s+4)x}\, dx$$

$$= C + \frac{6}{s+2} e^{-(2s+4)x}$$

Or, $U = Ce^{-(2s+1)x} + \frac{6}{s+2} e^{-3x}$

Where C is the constant of integration. Since when $x\to\infty$, u(x, t) must be bounded.

Therefore, when $x\to\infty$, U(x, t) should also be bounded

Hence, C = 0

The solution is $U = \frac{6}{s+2} e^{-3x}$

Now taking inverse Laplace transform of both the sides of

$$L^{-1}\{U\} = L^{-1}\left\{ \frac{6}{s+2} e^{-3x} \right\}$$

Therefore the required solution is $u(x, t) = \frac{6}{s+2} e^{-2t-3x}$

APPLICATIONS

1. Converting ODE into Algebraic Equation :

The amazing thing about using Laplace transforms is that we can convert the whole ODE initial value problem (I.V.P) into Laplace transformed functions of s, simplify the algebra, find transformed solution F(s). Then undo the transform to get back to the required solution f as a function of t.

Initial Value → Laplace Transform → Algebraic Equation

Example: Initial value problem: the basic Laplace steps

Solve $y'' - y = t, \quad y(0) = 1, \quad y'(0) = 1$

Given Problem

$y'' - y = t$

$y(0) = 1$

$y'(0) = 1$

→

Subsidiary Equation

$(s^2 - 1)\,Y = s + 1 + 1/s^2$

↓

Solution of Subsidiary Equation

$Y = 1/s-1 + 1/s^2 -1 - 1/s^2$

←

Solution of given problem

$Y(t) = e^t + \sinh t - t$

2. In Probability Theory :

It is used in probability theory to derive the moment-generating function of a probability distribution. The moment- generating function is used to find moments of a distribution, which are useful in statistical analysis.

3. In Economics :

Laplace transform helps in analyze economic systems, including the behavior of variables like GDP, inflation, and unemployment. It is used to study dynamic economic systems, enabling economists to understand how systems change over time. Used in econometrics to filter data and separate signal from noise.analyze the stability of economic systems. Helps economists understand economic cycles and oscillations, enabling better policy decisions.

4. In Biology :

- Modeling population dynamics
- Understanding pharmacokinetics: Laplace transform is used to study the absorption, distribution, metabolism, and excretion of drugs.
- Analyzing epidemiological models : It analyze the spread of diseases and understand the impact of interventions
- Modeling ecological systems: used to study the dynamics of ecological systems, including predator-prey relationships.
- Analyzing gene regulatory networks: It helps understand how genes are regulated and how their expression changes over time.
- Understanding the immune system_: Laplace transform helps analyze the behavior of the immune system and understand how it responds to pathogens.

5. Analyzing system :

(i) It is widely used to analyze and design <u>control systems</u>. It helps to convert time-domain signals into frequency-domain signals, making it easier to analyze and design the system's behaviour.

(ii) It is used to analyze and design <u>electrical circuits</u>. In addition, it helps to solve differential equations related to circuits and determine their stability and transient response.

(iii) It is used in mechanics to analyze the behaviour of mechanical systems, such as <u>structures' vibrations</u>, the <u>pendulum's motion</u>, and system dynamics.

6. In Finance :

In the field of finance, the Laplace transform is used to analyze financial time series, model derivative contracts, and study risk management problems and option pricing.

ADVANTAGES

1. Simplifies differential equations into algebraic equations.
2. Converts integral equations into algebraic equations.
3. Helps in stability analysis and determining system stability.
4. Facilitates transfer function analysis and obtaining transfer functions.
5. Enables filter design and analysis.
6. Analyzes electrical circuits and provides insight into circuit behavior.
7. Solves boundary value problems and partial differential equations.
8. Provides a powerful tool for solving PDEs and analyzing systems.
9. Facilitates control system design and ensures stability and desired performance.
10. Offers a unified approach to solving various types of equations and analyzing different systems.

LIMITATIONS

1. Limited to linear systems: Laplace transform is only applicable to linear systems, i.e limiting its use in analyzing nonlinear systems, which is common in many real-world application.

2. Requires initial conditions: Laplace transform requires initial conditions to be specified, which may not always be available or easy to determine.

3. Limited to time-invariant systems: It is only applicable to time-invarient system, i.e. you cannot use it to analyze systems that change over time, such as those with time-varying parameter.

4. Discontinuities: The Laplace transform may encounter difficulties in handling functions with discontinuities, such as step functions or impulses

5. Difficulty in dealing with non-integer order systems: Laplace transform is not well-suited for analyzing systems with non-integer order derivatives or integral.

6. Limited to systems with constant coefficients: Laplace transform assumes constant coefficients, which may not be the case in many real-world applications.

7. May not be suitable for systems with distributed parameters: Laplace transform is not well-suited for analyzing systems with distributed parameters.

Despite these disadvantages, the Laplace transform remains a powerful tool in many fields, and its advantages often outweigh its limitations. However, it's essential to be aware of these disadvantages choose the appropriate analytical tool for a specific problem.

CONCLUSION

The Laplace transformation is a powerful mathematical tool that has proven to be essential in many fields of study. For example, its ability to transform a function of time into a function of s has made it an invaluable tool for solving differential equations and analyzing linear time-invariant systems. The real-life applications of the Laplace transformation are vast and diverse, ranging from electrical circuits and control systems to economics and physics.

In this project, we explored the properties and applications of the Laplace transform.: Through this project, We learned how to apply the Laplace transform to various problems & gained a deeper understanding of the Laplace transform's ability to:

- Solving ordinary differential equations (ODEs)
- Solving partial differential equations (PDEs)
- Convert differential equations into algebraic equations
- Enable the use of algebraic methods to solve problems

We also saw how the Laplace transform is used in various fields, including physics, engineering, and economics. Overall, this project demonstrated the importance and versatility of the Laplace transform in solving real-world problems.

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